A Note on the \(w\)-Global Transform of Mori Domains

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Abstract Let \(R\) be a domain and let \(R^{wg}\) be the \(w\)-global transform of \(R\). In this note it is shown that if \(R\) is a Mori domain, then the \(t\)-dimension formula \(t\)-dim\((R^{wg}) = t\)-dim\((R) - 1\) holds.

Keywords Mori domain; maximal \(w\)-ideal; the \(w\)-global transform.

Throughout this paper \(R\) denotes a domain with quotient field \(K\). Matijevic in [6] had introduced the notion of the global transform of \(R\), which is defined to be the set

\[ R^g = \{ x \in K \mid M_1 \cdots M_n x \subseteq R, \text{ where } M_i \in \text{Max}(R) \}, \]

and shown that if \(R\) is Noetherian, then any ring \(T\) such that \(R \subseteq T \subseteq R^g\) is Noetherian. We have known that Mori domains have the ascending chain condition on divisorial ideals and strong Mori domains have the ascending chain condition on \(w\)-ideals. Every strong Mori domain is a Mori domain, but a Mori domain is not necessarily a strong Mori domain. Park [7] proved the \(w\)-analogue of Matijevic result, that is, if \(R\) is a strong Mori domain, then any \(w\)-overring \(T\) in the \(w\)-global transform \(R^{wg}\) of \(R\) is also a strong Mori domain. In this note we give the relationship of \(t\)-dimension of \(R\) and \(R^{wg}\) for a Mori domain \(R\).

Let \(A\) be a fractional ideal of \(R\). Define \(A^{-1} = \{ x \in K \mid xA \subseteq R \}\) and set \(A_v = (A^{-1})^{-1}\). If \(A = A_v\), then \(A\) is called a \(v\)-fractional ideal. We also define \(A_t = \bigcup B_v\), where \(B\) ranges over finitely generated fractional subideal of \(A\). If \(A_t = A\), then \(A\) is called a \(t\)-fractional ideal. Let \(J\) be a finitely generated ideal of \(R\). \(J\) is called a \(GV\)-ideal, denoted by \(J \in GV(R)\), if \(J^{-1} = R\). Define

\[ A_w = \{ x \in K \mid Jx \subseteq A \text{ for some } J \in GV(R) \}. \]

If \(A_w = A\), then \(A\) is called a \(w\)-fractional ideal, equivalently, the condition \(x \in K\) and \(J \in GV(R)\) with \(Jx \subseteq A\) implies \(x \in A\). For the discussion on \(t\)-ideals and \(w\)-ideals, readers can consult the

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literature [5] and [10]. Let $R \subseteq T$ be an extension of domains. We say that $T$ is an overring of $R$ if $T \subseteq K$. Let $T$ be an overring of $R$. Following [11] and [7], we call $T$ a $w$-overring if $T$ as an $R$-module is a $w$-module.

Let $P$ be a prime $t$-ideal of $R$. We denote by $t$-ht $P$ the supremum of the lengths $n$ of all chains $0 \subseteq P_n \subseteq P_{n-1} \subseteq \cdots \subseteq P_1 = P$, where $P_1, \ldots, P_{n-1}, P_n$ are prime $t$-ideals of $R$. Define $t$-$\dim(R) = \sup\{t$-ht $P\}$, where $P$ ranges over all prime $t$-ideals of $R$.

Denote by $w$-$\text{Max}(R)$ the set of maximal $w$-ideals of $R$. Following the notation of Park [7], we denote

$$R^{wg} = \{x \in K \mid P_1 \cdots P_n x \subseteq R \text{ for some } P_1, \ldots, P_n \in w$-$\text{Max}(R)\}.$$ 

Then $R^{wg}$ is an overring of $R$ contained in $K$ and is called $w$-global transform of $R$.

Let $B$ be a fractional ideal of $R$. Define similarly the $w$-global transform of $B$ to be the set

$$B^{wg} = \{x \in K \mid P_1 \cdots P_n x \subseteq B \text{ for some } P_1, \ldots, P_n \in w$-$\text{Max}(R)\}.$$ 

**Lemma 1** $R^{wg}$ is a $w$-overring of $R$.

**Proof** See [7, Corollary 1.7].

**Lemma 2** (1) Let $B_1$ and $B_2$ be fractional ideals of $R$ with $B_1 \subseteq B_2$. Then $(B_1)^{wg} \subseteq (B_2)^{wg}$.

(2) Let $B$ be a fractional ideal of $R$. Then $B^{wg}$ is a fractional ideal of $R^{wg}$.

(3) If $B$ is an ideal of $R$, then $B^{wg} = R^{wg}$ if and only if there are $P_1, \ldots, P_n \in w$-$\text{Max}(R)$ such that $P_1 \cdots P_n \subseteq B$. Therefore, if $Q$ is a prime ideal of $R$, then $Q^{wg} = R^{wg}$ if and only if $P \subseteq Q$ for some $P \in w$-$\text{Max}(R)$.

(4) Let $A$ be an ideal of $R^{wg}$ and let $B = A \cap R$. Then $A \subseteq B^{wg}$.

**Proof** It is straightforward.

**Lemma 3** (1) Let $Q$ be a prime ideal of $R$ such that $P \not\subseteq Q$ for any $P \in w$-$\text{Max}(R)$. Then $Q^{wg}$ is a prime ideal of $R^{wg}$ and $Q^{wg} \cap R = Q$.

(2) Let $Q_1$ and $Q_2$ be prime ideals of $R$ with $P \not\subseteq Q_1, Q_2$ for any $P \in w$-$\text{Max}(R)$. Then $(Q_1)^{wg} = (Q_2)^{wg}$ if and only if $Q_1 = Q_2$.

(3) Let $A$ be a prime ideal of $R^{wg}$ and let $Q = A \cap R$. If $P \not\subseteq Q$ for any $P \in w$-$\text{Max}(R)$, then $A = Q^{wg}$.

(4) Let $Q$ be a prime ideal of $R$ such that $P \not\subseteq Q$ for any $P \in w$-$\text{Max}(R)$. Then $\text{ht} Q^{wg} = \text{ht} Q$.

**Proof** (1) By Lemma 2, $Q^{wg} \neq R^{wg}$. Let $x, y \in R^{wg}$ with $xy \in Q^{wg}$. Then there are $P_1, \ldots, P_n, P_{n+1}, \ldots, P_m \in w$-$\text{Max}(R)$ such that $P_1 \cdots P_n x \subseteq R$, $P_{n+1} \cdots P_m y \subseteq R$, and $P_1 \cdots P_n P_{n+1} \cdots P_m x y \subseteq Q$. Hence $P_1 \cdots P_n x \subseteq Q$ or $P_{n+1} \cdots P_m y \subseteq Q$, that is, $x \in Q^{wg}$ or $y \in Q^{wg}$. Then $Q^{wg}$ is a prime ideal of $R^{wg}$.

It is clear that $Q \subseteq Q^{wg} \cap R$. Conversely, let $a \in Q^{wg} \cap R$. Then $P_1 \cdots P_n a \subseteq Q$ for $P_1, \ldots, P_n \in w$-$\text{Max}(R)$. Since $P_1 \not\subseteq Q$, we have $a \in Q$. Hence $Q = Q^{wg} \cap R$.

(2) If $(Q_1)^{wg} = (Q_2)^{wg}$, then $Q_1 = (Q_1)^{wg} \cap R = (Q_2)^{wg} \cap R = Q_2$. 

Lemma 4
(1) Let $B$ be a fractional ideal of $R$. Then, as fractional ideals of $R^w$, $(B^{-1})^w \subseteq (B^w)^{-1}$. 

(2) Let $B$ be a $t$-finite type fractional ideal of $R$. Then $(B^{-1})^w = (B^w)^{-1} = (BR^w)^{-1}$. 

(3) Let $R$ be a Mori domain and let $B$ be a fractional ideal of $R$. Then, as fractional ideals of $R^w$, $(B^w)_v = (BR^w)_v = (B_v)^w$. Therefore, if $B$ is a $v$-ideal of $R$, then $B^w$ is a $v$-ideal of $R^w$. 

(4) Let $R$ be a Mori domain and let $A$ be an ideal of $R^w$. Then $A_v = (B_v)^w$, where $B = A \cap R$. Therefore, if $A$ is a $v$-ideal of $R^w$, then $B = A \cap R$ is a $v$-ideal of $R$ and $A = B^w = (BR^w)_v$. 

(5) Let $R$ be a Mori domain and let $B$ be an ideal of $R$. Then $(B^w)^{-1} = R^w$ if and only if there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n \subseteq B_v$. Therefore, $(PR^w)^{-1} = R^w$ for any $P \in w\text{-Max}(R)$. 

(6) Let $R$ be a Mori domain and let $A$ be an ideal of $R^w$. Then $A_v = R^w$ if and only if there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n \subseteq B_v$, where $B = A \cap R$.

Proof
(1) Let $x \in (B^{-1})_x$. There are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n x \subseteq B^{-1}$. For any $y \in B^w$, take $P_{n+1}, \ldots, P_m \in w\text{-Max}(R)$ such that $P_{n+1} \cdots P_m y \subseteq B$. Thus $P_1 \cdots P_m xy \subseteq B^{-1} B \subseteq R$. Hence $xy \in R^w$. Thus $x \in (B^w)^{-1}$, whence, $(B^{-1})^w \subseteq (B^w)^{-1}$. From $BR^w \subseteq B^w$, we have $(BR^w)^{-1} \subseteq (B^w)^{-1}$.

(2) It suffices by (1) to show that $(BR^w)^{-1} \subseteq (B^{-1})^w$. Let $x \in (BR^w)^{-1}$. Since $B$ is of $t$-finite type, there is a finitely generated fractional subideal $J$ of $B$ such that $B_v = J_v$, therefore, $J^{-1} = B^{-1}$. Because $xJ \subseteq xB \subseteq R^w$ and $J$ is finitely generated, there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n J x \subseteq R$. Then $P_1 \cdots P_n x \in J^{-1} = B^{-1}$. Hence $x \in (B^{-1})^w$. Thus we have $(BR^w)^{-1} \subseteq (B^{-1})^w$.

(3) This follows from (2) since $B^{-1}$ is also of $t$-finite type in a Mori domain.

(4) Since $BR^w \subseteq A \subseteq B^w$ by Lemma 2 (4), we have $(BR^w)_v \subseteq A_v \subseteq (B^w)_v$. Hence $A = (B_v)^w$ by (3).

Suppose $A$ is a $v$-ideal of $R^w$. Since $BR^w \subseteq A \subseteq B^w$, we have $(BR^w)_v \subseteq A \subseteq (B_v)^w$. Hence $A = (BR^w)_v = (B_v)^w$. Then $B_v \subseteq A \cap R = B$, that is, $B = B_v$. Hence $A = B^w = (BR^w)_v$.

(5) From (3), $(B^w)^{-1} = R^w$ if and only if $(B_v)^w = R^w$, if and only if there are $P_1, \ldots, P_n \in w\text{-Max}(R)$ such that $P_1 \cdots P_n \subseteq B_v$ by Lemma 2.

(6) It is direct from (4) and (5). □

Proposition 5
Let $R$ be a Mori domain and let $A$ be a $w$-ideal of $R^w$. Then $B = A \cap R$ is a...
w-ideal of $R$ and $A = B^{w^g} = (BR^{w^g})_w$.

**Proof** By [11, Lemma 3.1], $B$ is a $w$-ideal of $R$. Since $B \subseteq A$, we have $BR^{w^g} \subseteq A \subseteq B^{w^g}$. Hence $(BR^{w^g})_w \subseteq A \subseteq B^{w^g}$. Let $x \in B^{w^g}$. Then there are $P_1, \ldots, P_n \in w$-$\text{Max}(R)$ such that $P_1 \cdots P_n x \subseteq B$. Let $I_i$ be a finitely generated subideal of $P_i$ such that $P_i = (I_i)_v$ for $i = 1, \ldots, n$. Thus $I_1 \cdots I_n x \subseteq B$. By Lemma 4, $I_1 R^{w^g} \in GV(R^{w^g})$. Then $x \in (BR^{w^g})_w$, and hence $A = (BR^{w^g})_w = B^{w^g}$. □

**Proposition 6** (1) Let $R$ be a Mori domain. Then $R^{w^g}$ is also a Mori domain.

(2) Let $R$ be a strong Mori domain. Then $R^{w^g}$ is also a strong Mori domain.

**Proof** (1) It follows from Lemma 4. Also see [8, Théorème 2].

(2) It follows from Proposition 5. Also see [7, Theorem 1.5 & Corollary 1.7]. □

**Theorem 7** Let $R$ be a Mori domain. Let $A$ be a maximal $v$-ideal of $R^{w^g}$ and set $B = A \cap R$. Then, for any $P \in w$-$\text{Max}(R)$, $P \not\subseteq B$, and $B$ is a maximal prime $v$-subideal of $P$ for any maximal $v$-ideal $P$ of $R$ with $B \subseteq P$.

**Proof** For any $P \in w$-$\text{Max}(R)$, then $P$ is a $v$-ideal because $R$ is a H-domain by [5]. Write $P = J_v$, where $J$ is a finitely generated subideal of $P$. By Lemma 4(6), $JR^{w^g} \in GV(R^{w^g})$. Hence $P \not\subseteq B$.

By Lemma 4, $B$ is a prime $v$-ideal of $R$ and $A = B^{w^g}$. Let $P$ be a maximal $w$-ideal of $R$ with $B \subseteq P$ and let $Q$ be a prime $v$-ideal of $R$ with $B \subseteq Q \subseteq P$. If $Q \neq P$, then $Q^{w^g}$ is a prime $v$-ideal of $R^{w^g}$ by Lemma 3 and Lemma 4. Hence $A = Q^{w^g}$ by the maximality of $A$. Then $B = Q$ by Lemma 3 again. □

**Theorem 8** Let $R$ be a Mori domain (but not a field). Then $t$-$\text{dim}(R^{w^g}) = t$-$\text{dim}(R) - 1$.

**Proof** Let $A_n \subseteq A_{n-1} \subseteq \cdots \subseteq A_1 \subseteq A_0$ be a chain of prime $v$-ideals of $R^{w^g}$. Set $B_i = A_i \cap R$ for $i = 0, 1, \ldots, n$. Then $B_i$ is a prime $v$-ideal of $R$ by Lemmas 3 and 4, and $B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1 \subseteq B_0$ be a chain of prime $v$-ideals of $R$. By Theorem 7, $B_0$ is not a maximal $t$-ideal of $R$. Hence $t$-$\text{dim}(R^{w^g}) \leq t$-$\text{dim}(R) - 1$. Conversely, let $B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1 \subseteq B_0$ be a chain of prime $v$-ideals of $R$ such that $B_0$ is not maximal $v$-ideal of $R$. By Lemma 3, $B_n^{w^g} \subset B_{n-1}^{w^g} \subset \cdots \subset B_1^{w^g} \subseteq B_0^{w^g}$ is a chain of prime $v$-ideals of $R^{w^g}$. Hence $t$-$\text{dim}(R^{w^g}) \geq t$-$\text{dim}(R) - 1$. □

**Corollary 9** Let $R$ be a Mori domain. If $t$-$\text{dim}(R) = 1$, then $R^{w^g} = K$.

**Proof** Since $t$-$\text{dim}(R) = 1$, we have $t$-$\text{dim}(R^{w^g}) = 0$ by Theorem 8. Hence $R^{w^g}$ is a field, that is, $R^{w^g} = K$. □

**References**

