On $X$-Semipermutability of Some Subgroups of Finite Groups

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Abstract Let $A$ be a subgroup of a group $G$ and $X$ a nonempty subset of $G$. $A$ is said to be $X$-semipermutable in $G$ if $A$ has a supplement $T$ in $G$ such that $A$ is $X$-permutable with every subgroup of $T$. In this paper, we try to use the $X$-semipermutability of some subgroups to characterize the structure of finite groups.

Keywords finite groups; $X$-semipermutable subgroups; supersoluble groups; $p$-nilpotent groups.

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1. Introduction

All groups in this paper will be finite.

Let $A$ and $B$ be subgroups of a group $G$. $A$ is said to be permutable with $B$ if $AB = BA$. $A$ is said to be a permutable subgroup (or quasinormal subgroup) of $G$ if $A$ is permutable with all subgroups of $G$ (see [13]). The permutable subgroups have many interesting properties. For example, Ore [13] proved that every permutable subgroup of a group $G$ is subnormal in $G$. Ito and Scép [10] proved that for every permutable subgroup $H$ of a group $G$, $H/H_G$ is nilpotent. However, two subgroups $H$ and $T$ of a group $G$ may not be permutable in $G$, but $G$ may contain an element $x$ such that $HT^x = T^xH$. Basing on the observation, Guo, Shum and Skiba [4, 5, 7] recently introduced the following generalized permutable subgroups. Let $A$ and $B$ be subgroups of a group $G$ and $X$ a nonempty subset of $G$. Then $A$ is said to be $X$-permutable with $B$ if there exists some $x \in X$ such that $AB^x = B^xA$; $A$ is said to be $X$-permutable (or $X$-quasinormal) in $G$ if for every subgroup $K$ of $G$ there exists some $x \in X$ such that $AK^x = K^xA$; $A$ is said to be $X$-semipermutable in $G$ if $A$ is $X$-permutable with all subgroups of some supplement $T$ of $A$ in $G$ (see [4]). By using the generalized permutable subgroups, one has obtained some important results [4–7, 9, 11, 12, 15].

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As a continuation, we try to use some $F(G)$-semipermutable subgroups to determine the structure of finite groups, where $F(G)$ is the Fitting subgroup of a group $G$. Some new interesting results are obtained.

For notations and terminologies not given in this paper, the reader is referred to [3] and [14].

2. Preliminaries

We cite here some known results which are useful in the sequel.

Lemma 2.1 ([4, Lemma 2.4]) Let $A$ and $X$ be subgroups of a group $G$. Then the following statements hold:

(1) If $N$ is a permutable subgroup of $G$ and $A$ is $X$-semipermutable in $G$, then $NA$ is an $X$-semipermutable subgroup of $G$.

(2) If $N$ is normal in $G$, $A$ is $X$-semipermutable in $G$ and $T$ is a supplement of $A$ in $G$ such that every subgroup of $T$ is $X$-permutable with $A$, then $AN/N$ is $X \cap N/N$-semipermutable in $G/N$ and $TN/N$ is a supplement of $AN/N$ in $G/N$ such that every subgroup of $TN/N$ is $X \cap N/N$-permutable with $AN/N$.

(3) If $A/N$ is $X \cap N/N$-semipermutable in $G/N$ and $T/N$ is a supplement of $A/N$ in $G/N$ such that every subgroup of $T/N$ is $X \cap N/N$-permutable with $A/N$, then $A$ is $X$-semipermutable in $G$ and $T$ is a supplement of $A$ in $G$ such that every subgroup of $T$ is $X$-permutable with $A$.

(4) If $A$ is $X$-semipermutable in $G$ and $A \leq D \leq G$, $X \leq D$, then $A$ is $X$-semipermutable in $D$.

(5) If $A$ is a maximal subgroup of $G$, $T$ is a minimal supplement of $A$ in $G$ and every subgroup of $T$ is $G$-permutable with $A$, then $T = \langle a \rangle$ is a cyclic $p$-subgroup for some prime $p$ and $a^p \in A$.

(6) If $A$ is $X$-semipermutable in $G$, $T$ is a supplement of $A$ in $G$ such that $A$ is $X$-permutable with every subgroup of $T$ and $A \leq N_G(X)$, then $T^x$ is a supplement of $A$ in $G$ such that every subgroup of $T^x$ is $X$-permutable with $A$, for all $x \in G$.

(7) If $A$ is $X$-semipermutable in $G$ and $X \leq D$, then $A$ is $D$-semipermutable in $G$.

Lemma 2.2 ([8, Lemma 2.6]) Let $H$ be a nilpotent normal subgroup of a group $G$. If $H \neq 1$ and $H \cap \Phi(G) = 1$, then $H$ has a complement in $G$ and $H$ is a direct product of some minimal normal subgroups of $G$.

Lemma 2.3 ([11, Lemma 3.3]) Let $G$ be a group and $X$ a normal $p$-soluble subgroup of $G$. Then $G$ is $p$-soluble if and only if a Sylow $p$-subgroup $P$ of $G$ is $X$-permutable with all Sylow $q$-subgroups of $G$, where $q \neq p$.

Lemma 2.4 Let $G$ be a soluble group and $N$ a minimal normal subgroup of $G$. If every minimal subgroup of $N$ is $G$-semipermutable in $G$, then $N$ is cyclic of prime order.

Proof Obviously $N$ is an elementary abelian $p$-group for some prime $p$. By Jordan-Hölder theorem, we can choose a minimal subgroup $A$ of $N$ such that $A$ is normal in some Sylow $p$-
subgroup $P$ of $G$. By the hypothesis, $A$ has a supplement $T$ in $G$ such that every subgroup of $T$ is $G$-permutable with $A$. Let $D$ be a Hall $p'$-subgroup of $T$. Then $E = AD^x = D^xA$ for some $x \in G$. Since $A$ is subnormal in $G$, $A$ is subnormal in $E$. Then since $D^x$ is a Hall $p'$-subgroup of $G$, $D^x \leq N_G(A)$. It follows that $G \leq N_G(A)$. The minimal choice of $N$ implies that $N = A$ and thereby $N$ is cyclic of order $p$.

### 3. Main results

**Theorem 3.1** Let $G$ be a group. Then $G$ is supersoluble if the normalizer of every Sylow subgroup of $G$ is $F(G)$-semipermutable in $G$.

**Proof** We first prove that $G$ is soluble. Let $p$ be the maximal prime dividing $|G|$ and $G_p$ a Sylow $p$-subgroup of $G$. Assume that $G_p$ is normal in $G$ and $Q/G_p$ is a Sylow $q$-subgroup of $G/G_p$, where $q \neq p$. Then $Q/G_p = G_qG_p/G_p$ for some Sylow $q$-subgroup $G_q$ of $G$ and $N_G/G_p(Q/G_p) = N_G(G_q)G_p/G_p$. By Lemma 2.1, we see that $N_G(G_p)(Q/G_p)$ is $F(G/G_p)$-permutable in $G/G_p$. This shows that $G/G_p$ satisfies the hypothesis. Hence $G/G_p$ is soluble by induction on $|G|$. It follows that $G$ is soluble. Now assume that $N_G(G_p) < G$. By hypothesis, $N_G(G_p)$ is $F(G)$-semipermutable in $G$ and so there exists a supplement $T$ of $N_G(G_p)$ in $G$ such that $N_G(G_p)$ is $F(G)$-permutable with every subgroup of $T$. Let $T_q$ be any Sylow $q$-subgroup of $T$, where $q \neq p$. Then $N_G(G_p)T_q = T_qN_G(G_p)$ for some $x \in F(G)$. Put $L = N_G(G_p)T_q^x$. Then $|L : N_G(G_p)| = q^β$. We claim that there exists some $q \neq p$ such that $β ≠ 0$. If not, then $N_G(G_p) = G$, a contradiction. Assume that $β > 1$ and $T_1$ is a maximal subgroup of $T_q$. Then there exists some $y \in F(G)$ such that $L_1 = N_G(G_p)T_1^y$ is a subgroup of $G$. Let $|L_1 : N_G(G_p)| = q^β_1$, where $β_1 ≤ β$. If $β_1 ≠ 1$, then by the same argument we see that there exists an $r$-maximal subgroup $T_r$ of $T_q$ such that $|L_r : N_G(G_p)| = q$, where $L_r = N_G(G_p)T_r^z$ for some $z \in F(G)$. Therefore $L_r$ has exactly $q$ Sylow $p$-subgroups. By Sylow’s theorem, $p$ divides $q - 1$, which is impossible since $p > q$. This contradiction shows that $N_G(G_p) = G$ and so $G$ is soluble.

Now we prove that $G$ is supersoluble by induction on $|G|$. Let $N$ be a minimal normal subgroup of $G$ and $P/N$ a Sylow $p$-subgroup of $G/N$, where $p$ divides $|G|$. Then $P/N = G_pN/N$ for some Sylow $p$-subgroup $G_p$ of $G$. Hence $N_G/N(P/N) = N_G(G_p)/N/N$. Since $N_G(G_p)$ is $F(G)$-permutable in $G$, $N_G/N(P/N)$ is $F(G/N)$-permutable in $G/N$. Hence $G/N$ is supersoluble by induction. Since the class of all supersoluble groups is a saturated formation, $N$ is the unique minimal normal subgroup of $G$ and $Φ(G) = 1$. Let $M$ be a maximal subgroup of $G$ such that $N$ is not contained in $M$. Then $G = [N]M$ and $N = C_G(N) = F(G) = O_q(G)$ for some prime $q$. Let $s$ be the largest prime dividing $|M|$ and $M_s$ a Sylow $s$-subgroup of $M$. Then $N_G(M_s) = M$ since $M$ is supersoluble and since $N$ is the unique minimal normal subgroup of $G$. If $s ≠ q$, then $M_s$ is a Sylow $s$-subgroup of $G$. By hypothesis and Lemma 2.1(5), we have that $[G : M] = q$ and so $|N| = q$, which implies that $G$ is supersoluble. If $s = q$, then $O_q(G/C_G(N)) = O_q(G/N) ≠ 1$, which contradicts [3, Lemma 1.7.11]. Then the proof is thus completed. □

**Corollary 3.2** Let $G$ be a group. If the normalizer of every Sylow subgroup of $G$ is $F(G)$-
quasinormal in $G$, then $G$ is supersoluble.

**Theorem 3.3** Let $G$ be a group and $M$ a supersoluble maximal subgroup of $G$. If $M$ is $F(G)$-semipermutable in $G$ and $F(G)$ is not contained in $M$, then $G$ is supersoluble.

**Proof** If $\Phi(G) \not= 1$, then by Lemma 2.1 we easily see that $G/\Phi(G)$ satisfies the hypothesis and so $G/\Phi(G)$ is supersoluble by induction on $|G|$. This implies that $G$ is supersoluble. Now assume that $\Phi(G) = 1$. By Lemma 2.2, $F(G)$ is a direct product of some minimal normal subgroups of $G$. Since $F(G)$ is not contained in $M$, $G = [N]M$, where $N$ is a minimal normal subgroup of $G$ contained in $F(G)$. By Lemma 2.1(5), we have that $|G : M| = p$ for some prime $p$. Since $G/N \cong M$ is supersoluble and $|N| = |G : M| = p$, we obtain that $G$ is supersoluble. \(\blacksquare\)

**Theorem 3.4** Let $p$ be an odd prime dividing the order of a group $G$, $P$ a Sylow $p$-subgroup of $G$ and $X = O_{p'}(G)$. If $N_G(P)$ is $p$-nilpotent and every cyclic subgroup of $P$ is $X$-semipermutable in $G$, then $G$ is $p$-nilpotent.

**Proof** Assume that the assertion is false and let $G$ be a counterexample of minimal order. Then we have the following claims:

1. Assume that $O_{p'}(G) \not= 1$. Then by using Lemma 2.1, we see that $G/O_{p'}(G)$ satisfies the hypothesis. Thus $G/O_{p'}(G)$ is $p$-nilpotent by the choice of $G$. Consequently $G$ is $p$-nilpotent, a contradiction.

2. If $M$ is a proper subgroup of $G$ such that $P \leq M \leq G$, then $M$ is $p$-nilpotent.

3. Assume that $O_p(G) = 1$ and $\{x_1, x_2, \ldots, x_n\}$ is a minimal generator set of $P$. Then by the hypothesis, for every $i$, $\langle x_i \rangle$ has a supplement $T_i$ in $G$ such that $\langle x_i \rangle$ is permutable with every subgroup of $T_i$. By Lemma 2.1(6), we see that $\langle x_i \rangle$ is permutable with every Sylow $q$-subgroup of $G$, where $q \not= p$. Then since $P = \langle x_1, x_2, \ldots, x_n\rangle$, $P$ is permutable with every Sylow $q$-subgroup of $G$, where $q \not= p$. It follows from Lemma 2.3 that $G$ is $p$-soluble and thereby $O_{p'}(G) > 1$ by (1), a contradiction. Hence (3) holds.

4. $G = PQ$, where $Q$ is a Sylow $q$-subgroup of $G$ and $q \not= p$.

Clearly the hypothesis still holds on $G/O_p(G)$. Hence $G/O_p(G)$ is $p$-nilpotent by the choice of $G$. Then by [2, Ch. 6, Theorem 3.5], for any prime $q$ dividing the order of $G$ with $q \not= p$, there exists a Sylow $q$-subgroup of $G$ such that $E = PQ$ is a subgroup of $G$. If $E < G$, then $E$ is $p$-nilpotent by (2). This leads to that $Q \leq C_G(O_p(G)) \leq O_p(G)$ (see [3, Theorem 1.8.19]), a contradiction. Thus $G = PQ$.

5. Final contradiction.

Let $N$ be a minimal normal subgroup of $G$. Then $N$ is an elementary abelian $p$-group because $G$ is soluble by (4) and $O_{p'}(G) = 1$. It is easy to see that $G/N$ satisfies the hypothesis. Hence
G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is the only minimal normal subgroup of G and Φ(G) = 1. Thus G = [N]M for some maximal subgroup M of G and N = C_G(N) = F(G) = O_p(G) = X. By Lemma 2.4, N is cyclic of order p. Since M ≅ G/N = G/C_G(N) is isomorphic to some subgroup of Aut(N), M is a p'-subgroup of G and so N is a Sylow p-subgroup of G. Then G = N_G(N) = N_G(P) is p-nilpotent by hypothesis. The final contradiction completes the proof. □

**Remark 3.5** The assumption “N_G(P) is p-nilpotent” in Theorem 3.4 is essential. For example, let G = S_3 and p = 3. Then every subgroup of Sylow 3-subgroup of G is O_p^2(G)-semipermutable in G, but G is not 3-nilpotent.

However, if p is the smallest prime dividing the order of G, then we have the following result.

**Theorem 3.6** Suppose that p is the smallest prime dividing the order of a group G and P is a Sylow p-subgroup of G. Let X = O_p^2(G). If every cyclic subgroup of P is X-semipermutable in G, then G is p-nilpotent.

**Proof** Assume that the assertion is false and let G be a counterexample of minimal order. Then

1. O_p^2(G) = 1.

Suppose that O_p^2(G) ≠ 1. Then it is easy to see that G/O_p^2(G) satisfies the hypothesis. The minimal choice of G implies that G/O_p^2(G) is p-nilpotent and so G is p-nilpotent, a contradiction.

2. O_p(G) ≠ 1.

Assume that O_p(G) = 1. Then X = 1 by (1). Let A be a minimal subgroup of P and T a supplement of A in G such that A is X-permutable with every subgroup of T. If A ∩ T = 1, then |G : T| = p and so T is normal in G (see [16, II, Proposition 4.6]). It is easy to see that T satisfies the hypothesis and hence T is p-nilpotent by the choice of G. If O_p^2(T) ≠ 1, then O_p^2(G) ≠ 1 because O_p^2(T) char T ≤ G. If O_p^2(T) = 1, then G is a p-group. This contradiction implies that A ∩ T = A. Thus G = T and so A is a permutable subgroup of G. By [14, (13.2.2)], A is subnormal in G and consequently A ≤ O_p(G) by [1, A, Lemma 8.6]. This contradiction shows that O_p(G) ≠ 1.

3. Φ(G) = 1.

By (1), we have F(G) = O_p(G). If Φ(G) ≠ 1, then Φ(G) ≤ O_p(G). Obviously, G/Φ(G) satisfies the hypothesis. Hence G/Φ(G) is p-nilpotent by the choice of G. It follows that G is p-nilpotent, a contradiction.

4. O_p(G) is a minimal normal subgroup of G.

By Lemma 2.2 and (3), O_p(G) is a direct product of some minimal normal subgroup of G. If O_p(G) is not a minimal normal subgroup of G, then obviously G/N satisfies the hypothesis for any minimal normal subgroup N of G contained in O_p(G). Hence G/N is p-nilpotent by choice of G. Since the class of all p-nilpotent groups is a saturated formation, O_p(G) must be a minimal normal subgroup of G.

5. The final contradiction.

If P is normal in G, then G/O_p(G) is a p'-subgroup and G is soluble. If P ≠ O_p(G), then
obviously $G/O_p(G)$ satisfies the hypothesis. Hence $G/O_p(G)$ is $p$-nilpotent. Let $K/O_p(G)$ be the normal $p$-complement of $P/O_p(G)$ in $G/O_p(G)$. Since $p$ is the smallest prime dividing the order of $G$, $K/O_p(G)$ is soluble by the well known Feit-Thompson theorem. It follows that $K$ is soluble and hence $G$ is soluble. Then by (4) and Lemma 2.4, $O_p(G)$ is cyclic of order $p$. Let $g$ be an element of $G$ such that $(|g|, p) = 1$ and $E = O_p(G)\langle g \rangle$. Then $\langle g \rangle$ is normal in $E$ by [14, (10.1.9)] and so $\langle g \rangle \leq C(G/O_p(G)) = O_p(G)$. The final contradiction completes the proof. \(\square\)

**Remark 3.7** The condition that “every cyclic subgroup of $P$ is $X$-semipermutable in $G$” in Theorem 3.6 cannot be replaced by “every minimal subgroup of $P$ is $X$-semipermutable in $G$”. For example, let $G = \langle a, b \rangle \langle \alpha \rangle$, where $a^4 = 1$, $a^2 = b^2 = [a, b]$ and $a\alpha = b$, $b\alpha = ab$. Then $G$ is a counterexample.

**References**