Maximal Armendariz Subrings Relative to a Monoid of Matrix Rings

Wen Kang WANG
School of Computer Science and Information Engineering, Northwest University for Nationalities, Gansu 730124, P. R. China

Abstract Let $M$ be a monoid. Maximal $M$-Armendariz subrings of upper triangular matrix rings are identified when $R$ is $M$-Armendariz and reduced. Consequently, new families of $M$-Armendariz rings are presented.

Keywords $M$-Armendariz ring; matrix ring; reduced ring.

Document code A
MR(2000) Subject Classification 16N60; 16P60
Chinese Library Classification O153.3

1. Introduction

Throughout this paper $R$ denotes an associative ring with identity. According to Rege and Chhawchharia [1], a ring $R$ is called Armendariz if, whenever $(\sum_{i=0}^{m} a_i x^i) (\sum_{j=0}^{n} b_j x^j) = 0$ in $R[x]$, $a_i b_j = 0$ for all $i$ and $j$. A ring is called reduced if it has no non-zero nilpotent elements. Every reduced ring is Armendariz by Armendariz [2], but the more comprehensive study of the notion of Armendariz rings was carried out just recently (see Anderson and Camillo [3], Kim and Lee [4], Hong, Kim and Kwak [5], Huh, Lee and Smoktunowicz [6], Lee and Wong [7], Lee and Zhou [8], Liu [9], Hong, Kim and Twak [10]).

According to Liu [11], a ring $R$ is called an $M$-Armendariz ring (an Armendariz ring relative to a monoid) if, whenever $(\sum_{i=1}^{m} a_i \alpha_i) (\sum_{j=1}^{n} b_j \beta_j) = 0$ in $R[M]$, $a_i b_j = 0$ for all $i$ and $j$.

In this paper, we continue the study of $M$-Armendariz rings, and focus on the $M$-Armendariz property of certain subrings of upper triangular matrix rings.

We denote by $T_n(R)$ and $M_n(R)$ the $n \times n$ upper triangular matrix ring and matrix ring over $R$, respectively.

Let $R$ be a ring. Define a subring $F_n$ of upper triangular matrix ring $T_n(R)$ over $R$ as follows:

$$F_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$
Let $M$ be a monoid. It was proved in Liu [11], Proposition 1.7 and Remark 1.8 that if $R$ is $M$-Armendariz and reduced, then the ring $F_3$ is $M$-Armendariz but $F_n$ is not $M$-Armendariz for $n ≥ 4$.

So, for an $M$-Armendariz and reduced ring $R$, it is interesting to find some maximal $M$-Armendariz subrings of $T_n(R)$. For this purpose, we define $S_{n,m}(R) =$

$$S_{n,m}(R) = \begin{pmatrix}
    a_1 & a_2 & \cdots & a_{m-1} & a_m & a_{1,m+1} & \cdots & a_{1,n-1} & a_{1,n} \\
    0 & a_1 & \cdots & a_{m-1} & a_{2,m+1} & \cdots & a_{2,n-1} & a_{2,n} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & 0 & a_1 & a_{m-1,m+1} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\
    0 & \cdots & 0 & a_1 & a_{m-1,m+1} & \cdots & a_{m-1,n-1} & a_{m-1,n} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & 0 & \cdots & a_1 & a_{m-1,m+1} & \cdots & a_{m-1,n-1} \\
0 & \cdots & 0 & \cdots & a_1 & a_{m-1,m+1} & \cdots & a_{m-1,n-1} \\
\end{pmatrix},$$

where $a_{i,j}, a_{i,j} ∈ R$, and show that for any $M$-Armendariz and reduced ring $R$ and all $2 ≤ m ≤ n - 1$, the ring $S_{n,m}(R)$ is a maximal $M$-Armendariz subring of $T_n(R)$. This is a generalization of Liu [11], Proposition 1.7 and Remark 1.8.

By the term “ring” we mean an associative ring with identity, and by a general ring we mean an associative ring with or without identity. For clarity, $R$ will always denote a ring while a general ring will be denoted by $I$.

Let $I$ be a general ring. Define a subring $D_n(I)$ of matrix ring $M_n(I)$ over $I$ as follows:

$$D_n(I) = \left\{ \begin{pmatrix}
    a_1 & a_1 & \cdots & a_1 \\
    a_2 & a_2 & \cdots & a_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    a_n & a_n & \cdots & a_n \\
\end{pmatrix} | a_i ∈ I \right\} \text{ for } n ≥ 2.$$}

We show that for any general $M$-Armendariz and reduced ring $I$, and $|M| ≥ 2$, $|I| ≥ 2$, the general ring $D_n(I)$ is a maximal general $M$-Armendariz subring of $M_n(I)$ for $n ≥ 2$.

### 2. Maximal $M$-Armendariz subrings of $T_n(R)$

**Lemma 2.1** Let $M$ be a monoid with $|M| ≥ 2$. If $R$ is $M$-Armendariz and reduced, then $R[M]$ is reduced.

**Proof** The proof has been shown in the proof of Liu [11, Proposition 2.1]. $\square$

According to [8], for $A = (a_{i,j})$, $B = (b_{i,j}) ∈ M_n(R)$, we write $[A \cdot B]_{i,j} = 0$ to mean that $a_{i,l}b_{l,j} = 0$ for $l = 1, \ldots, n$. We identify $M_n(R)[M]$ with $M_n(R[M])$ canonically.

For $n ≥ 2$, let $V = \sum_{i=1}^{n-1} E_{i,i+1}$ where $E_{i,j} : 1 ≤ i, j ≤ n$ are the matrix units.

**Lemma 2.2** Let $M$ be a monoid. For $u = \sum_{i=1}^{m} A_iα_i$, $v = \sum_{j=1}^{k} B_jβ_j ∈ M_n(R)[M]$, let
Since \( a_{ij}^{(s)} \alpha_s \) and \( g_{ij} = \sum_{l=1}^{k} b_{ij}^{(l)} \beta_l \) where \( a_{ij}^{(t)} \) and \( b_{ij}^{(h)} \) are the \((i, j)\)-entries of \( A_t \) and \( B_h \), respectively, for \( l = 1, \ldots, m, h = 1, \ldots, k \). Then \( u = (f_{ij}), v = (g_{ij}) \). If \( R \) is \( M \)-Armendariz and \( [u \cdot v]_{ij} = 0 \) for all \( i, j \), then \( A_iB_j = 0 \) for all \( i, j \).

**Proof** Since \( [u \cdot v]_{ij} = 0 \) for all \( i, j \) and \( R \) is \( M \)-Armendariz, \( a_{ii}^{(s)} b_{ij}^{(t)} = 0 \) for all \( i, j \), where \( l = 1, \ldots, n \). Then \( A_iB_j = 0 \) for all \( i, j \).

**Lemma 2.3** ([13, Theorem 2.3]) Let \( R \) be a reduced ring. If \( AB = 0 \) in \( S_{n,m}(R) \), then \( [A \cdot B]_{ij} = 0 \) for all \( i, j \) and all \( 2 \leq m \leq n \).

**Theorem 2.4** Let \( M \) be a monoid with \(|M| \geq 2\). Then the following conditions are equivalent.

1. \( R \) is \( M \)-Armendariz and reduced;
2. \( S_{n,m}(R) \) is an \( M \)-Armendariz ring for all \( 2 \leq m \leq n \).

**Proof** (1) \(\Rightarrow\) (2). Suppose that \( u = \sum_{s=1}^{p} A_s \alpha_s \), \( v = \sum_{j=1}^{p} B_j \beta_j \in S_{n,m}(R)[M] \) such that \( uv = 0 \). We need to prove that \( A_iB_j = 0 \) for all \( 1 \leq i, j \leq p \). Let \( f_{ij} = \sum_{s=1}^{p} a_{ij}^{(s)} \alpha_s \) and \( g_{ij} = \sum_{l=1}^{k} b_{ij}^{(l)} \beta_l \) where \( a_{ij}^{(t)} \) and \( b_{ij}^{(h)} \) are the \((i, j)\)-entries of \( A_t \) and \( B_h \), respectively, for \( l = 1, \ldots, p \). Then \( u = (f_{ij}), v = (g_{ij}) \). By Lemma 2.3, \( [u \cdot v]_{ij} = 0 \) for all \( i, j \), then \( A_iB_j = 0 \) for all \( 1 \leq i, j \leq p \) by Lemma 2.2.

(2) \(\Rightarrow\) (1). Suppose that \( S_{n,m}(R) \) is \( M \)-Armendariz. Note that \( R \) is isomorphic to the subring

\[
\begin{pmatrix}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\hline
0 & a & \cdots \\
\hline
0 & a & \cdots \\
\hline
\cdots & \cdots & \cdots \\
\hline
\end{pmatrix}
\]

of \( S_{n,m}(R) \). Thus \( R \) is \( M \)-Armendariz, since each subring of an \( M \)-Armendariz ring is also \( M \)-Armendariz. By analogy with the proof of Lee and Wong [7], Lemma 2.3, we can show that \( R \) is reduced.

**Corollary 2.5** ([11, Proposition 1.7]) Let \( M \) be a monoid with \(|M| \geq 2\). Then the following conditions are equivalent.

1. \( R \) is \( M \)-Armendariz and reduced;
2. \( S_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\} \) is \( M \)-Armendariz.

**Theorem 2.6** Let \( M \) be a monoid with \(|M| \geq 2\) and \( R \) be an \( M \)-Armendariz ring and reduced. Then \( S_{n,m}(R) \) is a maximal \( M \)-Armendariz subring of \( T_n(R) \) for all \( 2 \leq m \leq n - 1 \).

**Proof** Suppose that \( T \) is an \( M \)-Armendariz subring of \( T_n(R) \) and \( T \) properly contains \( S_{n,m}(R) \).

Take \( e \neq g \in M \), where \( e \) stands for the identity of \( M \). Then there exists \( A = (a_{ij}) \in T \) such that one of the following conditions holds:

1. \( a_{k-1,l-1} \neq a_{k,l} = 0 \) for some \( 2 \leq k \leq l \leq m \);
2. \( a_{k,l} \neq a_{k+1,l+1} \) for some \( m \leq k \leq l \leq n - 1 \).

**Case 1** Suppose that 1) holds. We can assume without loss of generality that \( a_{1,1+t} = a_{2,2+t} = \cdots = a_{m,} \)
$\cdots = a_{m-t,m}$ where $0 \leq t \leq l-k-1$, and $a_{1,l-k+1} = a_{2,l-k+2} = \cdots = a_{k-1,l-1}$. Let $V^0 = I_n$, and $A_1 = A - \sum_{t=0}^{l-k-1} a_{1,l-t+1} V^t - \sum_{t=0}^{l-k-1} a_{k-1,l-t} V^{l-k+t}$. Clearly $u = A_1 e - (a_{k-1,l-1} - a_{k,l}) V^{l-1} g \in T[M]$ and $v = E_{l,n} e + E_{l-k+1,n} g \in T[M]$. One easily checks that $uv = 0$ but $(a_{k-1,l-1} - a_{k,l}) V^{l-1} E_{l,n} = (a_{k-1,l-1} - a_{k,l}) E_{1,n} \neq 0$. Hence $T$ is not an $M$-Armendariz ring.

**Case 2** Suppose that 2) holds. We can assume without loss of generality that $a_{m,m+t} = a_{m+1,m+t+1} = \cdots = a_{n-t,n}$ where $0 \leq t \leq l-k-1$, and $a_{k+1,l+1} = a_{k+2,l+2} = \cdots = a_{k+t,n}$. Let $A_1 = A - \sum_{t=0}^{l-k-1} a_{m,m+t} V^t - \sum_{t=0}^{l-k-1} a_{k+1,l+t} V^{l-k+t}$. Then $u = (a_{k+1,l+1} - a_{k,l}) E_{1,k} e - E_{1,n-l+k} g$, $v = A_1 e + V^{n-k} g \in T[M]$. One checks that $uv = 0$, but $(a_{k+1,l+1} - a_{k,l}) E_{1,k} V^{n-k} = (a_{k+1,l+1} - a_{k,l}) E_{1,n} \neq 0$. Hence $T$ is not an $M$-Armendariz ring. □

By using the same methods as in the proof of Theorem 2.6, we have the following Theorem 2.7.

**Theorem 2.7** Let $M$ be a monoid with $|M| \geq 2$ and $R$ be an $M$-Armendariz ring and reduced. Then

$$F_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

is a maximal $M$-Armendariz subring of $T_2(R)$, and

$$F_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is a maximal $M$-Armendariz subring of $T_3(R)$.

3. Maximal general $M$-Armendariz subrings of $M_n(I)$

**Definition 3.1** A general ring $I$ is called general reduced if it has no non-zero nilpotent elements.

**Definition 3.2** Let $M$ be a monoid. A general ring $I$ is called general $M$-Armendariz if, whenever $(\sum_{i=1}^m a_i \alpha_i)(\sum_{j=1}^n b_j \beta_j) = 0$ in $I[M]$, $a_i b_j = 0$ for all $i$ and $j$.

Let $D_n(I) = \left\{ \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix} \mid a_i \in I \right\}$ for $n \geq 2$.

**Theorem 3.3** Let $M$ be a monoid. If $I$ is a general $M$-Armendariz ring, then $D_n(I)$ is a general $M$-Armendariz subring of $M_n(I)$ for $n \geq 2$.

**Proof** Suppose that $f = \sum_{i=1}^m A_i \alpha_i$, $g = \sum_{j=1}^n B_j \beta_j \in D_n(I)[M]$, such that $fg = 0$. We need to prove that $A_i B_j = 0$ for all $i$ and $j$.

Let

$$A_i = \begin{pmatrix} a_{1,i} & a_{1,i} & \cdots & a_{1,i} \\ a_{2,i} & a_{2,i} & \cdots & a_{2,i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,i} & a_{n,i} & \cdots & a_{n,i} \end{pmatrix}, \quad B_j = \begin{pmatrix} b_{1,j} & b_{2,j} & \cdots & b_{n,j} \\ b_{1,j} & b_{2,j} & \cdots & b_{n,j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1,j} & b_{2,j} & \cdots & b_{n,j} \end{pmatrix}$$
where $a_s^{(i)}, b_s^{(j)} \in I$ for $1 \leq s \leq n$ and $1 \leq i \leq m, 1 \leq j \leq k$, and let

$$f = \begin{pmatrix} f_1 & f_1 & \cdots & f_1 \\ f_2 & f_2 & \cdots & f_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_n & \cdots & f_n \end{pmatrix}, \quad g = \begin{pmatrix} g_1 & g_1 & \cdots & g_1 \\ g_2 & g_2 & \cdots & g_2 \\ \vdots & \vdots & \ddots & \vdots \\ g_n & g_n & \cdots & g_n \end{pmatrix},$$

where $f_v = \sum_{i=1}^m a_v^{(i)}a_i, g_v = \sum_{j=1}^k b_v^{(j)}b_j \in I$ for $1 \leq u, v \leq n$.

It follows from $fg = 0$ that

$$f_v[g_1 + g_2 + \cdots + g_n] = 0, \quad \text{for } 1 \leq u \leq n. \quad (3.1)$$

Because $I$ is a general $M$-Armendariz ring, we have

$$a_v^{(i)}[b_1^{(j)} + b_2^{(j)} + \cdots + b_n^{(j)}] = 0 \quad \text{for } 1 \leq i, j \leq m \text{ and } 1 \leq u \leq n. \quad (3.2)$$

Hence we show that $A_iB_j = 0$ for all $1 \leq i, j \leq m$. □

**Theorem 3.4** Let $M$ be a monoid and $|M| \geq 2$. If $I$ is a general $M$-Armendariz and reduced ring, and $|I| \geq 2$, then $D_n(I)$ is a maximal general $M$-Armendariz subring of $M_n(I)$ for $n \geq 2$.

**Proof** Suppose that $T$ is a general $M$-Armendariz subring of $M_n(I)$ and $T$ properly contains $D_n(I)$, then there exists $A = (a_{i,j}) \in T \setminus D_n(I)$ where $1 \leq i, j \leq n$. It suffices to show that $T$ is not general $M$-Armendariz. Take $e \neq g \in M$, where $e$ stands for the identity of $M$. We will proceed with the following two cases.

**Case 1** Suppose that $a_{11} = a_{12} = \cdots = a_{1,j-1} \neq a_{1,j}$ where $2 \leq j \leq n$. Then $a_{1,j-1} - a_{1,j} \neq 0$.

Let

$$A_1 = A - \begin{pmatrix} a_{1,j} & a_{1,j} & \cdots & a_{1,j} \\ a_{2,j} & a_{2,j} & \cdots & a_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j} & a_{n,j} & \cdots & a_{n,j} \end{pmatrix},$$

$$A_2 = A - \begin{pmatrix} a_{1,j-1} & a_{1,j-1} & \cdots & a_{1,j-1} \\ a_{2,j-1} & a_{2,j-1} & \cdots & a_{2,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j-1} & a_{n,j-1} & \cdots & a_{n,j-1} \end{pmatrix},$$

Then $A_1, A_2 \in T$. 


Let $f = A_1 e + A_2 g$ be in $T[M]$, and let

$$B_1 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
a_{1,j-1} - a_{1,j} & a_{1,j-1} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j}
\end{pmatrix} (j),$$

$$B_2 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
a_{1,j-1} - a_{1,j} & a_{1,j-1} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j}
\end{pmatrix} (j-1).$$

Then $B_1, B_2 \in T$.

Let $g = B_1 e + B_2 g$ be in $T[M]$. Then $fg = 0$, but

$$A_1 B_2 = \begin{pmatrix}
(a_{1,j-1} - a_{1,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{1,j-1} - a_{1,j})(a_{1,j-1} - a_{1,j}) \\
(a_{2,j-1} - a_{2,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{2,j-1} - a_{2,j})(a_{1,j-1} - a_{1,j}) \\
\vdots & \ddots & \vdots \\
(a_{n,j-1} - a_{n,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{n,j-1} - a_{n,j})(a_{1,j-1} - a_{1,j})
\end{pmatrix} \neq 0.$$

This is a contradiction.

**Case 2** Suppose that $a_{t,1} = a_{t,2} = \cdots = a_{t,n}$ where $1 \leq t \leq i - 1$, and $a_{i,1} = a_{i,2} = \cdots = a_{i,j-1} \neq a_{i,j}$ where $1 < i, j \leq n$. Then $a_{i,j-1} - a_{i,j} \neq 0$. Let

$$A_1 = A - \begin{pmatrix}
a_{11} & a_{11} & \cdots & a_{11} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i-1,i-1} & a_{i-1,i-1} & \cdots & a_{i-1,i-1} \\
a_{i,j} & a_{i,j} & \cdots & a_{i,j} \\
a_{i+1,j} & a_{i+1,j} & \cdots & a_{i+1,j} \\
\cdots & \cdots & \ddots & \cdots \\
a_{n,j} & a_{n,j} & \cdots & a_{n,j}
\end{pmatrix},$$

$$= \begin{pmatrix}
\cdots & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\
\cdots & 0 & 0 & 0 & \cdots & 0 \\
\cdots & a_{i,j-1} - a_{i,j} & 0 & a_{i,j-1} - a_{i,j} & \cdots & a_{i,j-1} - a_{i,j} \\
\cdots & a_{i+1,j-1} - a_{i+1,j} & 0 & a_{i+1,j-1} - a_{i+1,j} & \cdots & a_{i+1,j-1} - a_{i+1,j} \\
\cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\
\cdots & a_{n,j-1} - a_{n,j} & 0 & a_{n,j-1} - a_{n,j} & \cdots & a_{n,j-1} - a_{n,j}
\end{pmatrix},$$

where $1 \leq t \leq i - 1$, and $a_{i,1} = a_{i,2} = \cdots = a_{i,j-1} \neq a_{i,j}$.
Let $g = A_1 e + A_2 g$ be in $T[M]$, and let

$$B_1 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix} \quad (j)$$

$$B_2 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix} \quad (j - 1).$$

Then $B_1, B_2 \in T$.

Let $g = B_1 e + B_2 g$ be in $T[M]$. Then $fg = 0$, but

$$A_1 B_2 = \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 \\
\end{pmatrix} \neq 0.$$
This is a contradiction.

Thus $T$ is not general $M$-Armendariz. □

Let $D_n(I)^T = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} | a_i \in I \right\}$ for $n \geq 2$.

By using the same methods as in the proofs of Theorems 3.3 and 3.4, we have the following Theorem 3.5

**Theorem 3.5** Let $M$ be a monoid and $|M| \geq 2$. If $I$ is a general $M$-Armendariz and reduced ring, and $|I| \geq 2$, then $D_n^T(I)$ is a maximal general $M$-Armendariz subring of $M_n(I)$ for $n \geq 2$.

References