A Trace Theorem of Besov Spaces on a $d$-Set

Ming XU

Department of Mathematics, Ji’nan University, Guangdong 510632, P. R. China

Abstract In the paper we give a trace theorem of Besov spaces $B^{s,p}_p(\mathbb{R}^n)$ on a $d$-set. It is a kind of extension of related results of Jonsson and Wallin, which has important applications on PDE theory.

Keywords Besov spaces; trace; $d$-set.

MR(2000) Subject Classification 46E35; 42B35; 46F05

Chinese Library Classification O174.2

1. Introduction

Trace problem has many important applications in PDE theory [1]. The purpose of the paper is to study the trace theorem of Besov spaces $B^{s,q}_p$ for $p \leq 1$. It seems that in many references, most of them care about the problem when $p \geq 1$ (see [2, 3]). In [4], a trace theorem of Besov spaces for $p \leq 1$ was given on a Lipschitz domain without proof. In this paper, we mainly discuss the trace problem of Besov spaces for $p \leq 1$ on a $d$-set.

As we know, there are many examples known as $d$-sets, for example, a Lipschitz domain with a bounded Lipschitz boundary in $\mathbb{R}^n$ with $d = n - 1$ or a bounded hyperplane of $\mathbb{R}^n$ with $d < n$. For further examples, we refer to [2] and [5]. The $d$-set is defined as follows

Definition 1.1 Let $\Gamma$ be a closed, non-empty subset in $\mathbb{R}^n$, and $0 < d < n$. Then $\Gamma$ is called a $d$-set if there exists a Borel measure $\mu$ in $\mathbb{R}^n$ with the following properties:

1) $\operatorname{supp} \mu = \Gamma$;

2) for any $Q \in \Gamma$ and any $r$ with $0 < r < r_0$, there exist $c_1, c_2 > 0$ such that

$$c_1 r^d \leq \mu(B(Q,r) \cap \Gamma) \leq c_2 r^d,$$

where $B(Q,r)$ denotes the ball with center $Q \in \Gamma$ and radius $r$, and $r_0 > 0$ denotes the diameter of $\Gamma$.

Remark 1.1 Such definition can be found in [5, pp 1-5]. We also use “$A \sim B$” to denote “$c_1 B \leq A \leq c_2 B$” in the following.

Actually the trace theorem of Besov spaces on a $d$-set has been studied by several people. In [2, p. 103], Jonsson and Wallin proved that if $\Gamma$ is a $d$-set, then $B^{s,p}_p(\mathbb{R}^n)|_\Gamma = B^{s,p}_p(\Gamma)$ for

Received August 20, 2008; Accepted September 15, 2009

Supported by the Ji’nan University’s Young Scholar’s Foundation (Grant No. 51208036).

E-mail address: stxmin@163.com
\[ 1 \leq p \leq \infty, \quad 0 < \alpha < 1 \quad \text{and} \quad 0 < \beta = \alpha - \frac{n-d}{p} < 1. \]  
In [5, p.139], Triebel also gave the trace theorem that if \( \Gamma \) is a \( d \)-set, then \( B_p^{\frac{n-d}{p},q}(\mathbb{R}^n)|_{\Gamma} = L^p(\Gamma) \) for \( \frac{d}{n} < p < \infty, \quad 0 < q \leq \min(1,p) \). One can consider if we can obtain similar results for larger range to \( p \) and \( \alpha \).

The purpose of the paper is to prove

\textbf{Theorem 1.1} Let \( \Gamma \) be a \( d \)-set in \( \mathbb{R}^n \) (0 < \( d < n \)), \( \frac{d}{n} < p \leq \infty \) and \( \beta = \alpha - \frac{n-d}{p} < 1 + \frac{n-d}{p} \). Then

\[ B_p^{\alpha,p}(\mathbb{R}^n)|_{\Gamma} = B_p^{\beta,p}(\Gamma). \]

Throughout the paper, the letter “\( C \)” will denote (possibly different) constants that are independent of the essential variables.

\section{The proof of Theorem 1.1}

In the section, we prove the main theorem. Firstly we introduce the Besov spaces \( B_p^{\alpha,p}(\mathbb{R}^n) \) and \( B_p^{\alpha,p}(\Gamma) \). Take \( \varphi_0 \in \mathcal{S} \) and \( \psi(X) = \varphi_0(X) - 2^{-n}\varphi_0(X/2) \) satisfying

\[ \int_{\mathbb{R}^n} \varphi_0(X)dX = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} X^\alpha \psi(X)dX = 0, \quad \text{for any } |\alpha| \geq 0. \]

\textbf{Definition 2.1} Let \( \varphi_0 \in \mathcal{S} \) satisfy (2.1) and \( \psi(X) = \varphi_0(X) - 2^{-n}\varphi_0(X/2) \). Then for \( s \in \mathbb{R} \) and \( 0 < p \leq \infty, \quad 0 < q \leq \infty \)

\[ B_p^{s,q}(\mathbb{R}^n) \sim \{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{B_p^{s,q}(\mathbb{R}^n)} \}
\]

\[ = \{ t^{-\alpha} \| \psi_t * f \|_{L^p(\mathbb{R}^n)} \}^{1/\alpha} + \| \varphi_0 * f \|_{L^p(\mathbb{R}^n)} < \infty \}, \]

where \( \psi_t(X) = t^{-\alpha} \psi(X/t) \).

\textbf{Definition 2.2} Let \( \Gamma \) be a \( d \)-set (0 < \( d < n \)). For \( \frac{d}{n} < p \leq \infty \) and \( 0 \leq s < 1 \), the Besov space on \( \Gamma \) is defined as

\[ B_p^{s,p}(\Gamma) = \{ f \in L^p(\Gamma) : \| f \|_{L^p(\Gamma)} + \left( \int_{\{X,Y \in \Gamma : |X-Y| < 1 \}} \frac{|f(X) - f(Y)|^p}{|X-Y|^{d+sp}}d\mu(X)d\mu(Y) \}^{1/p} < +\infty \}.\]

Next we give the following lemma [3, p.58].

\textbf{Lemma 2.1} Let \( 0 < p \leq \infty, \quad s \in \mathbb{R} \) and \( \sigma \in \mathbb{R} \), for any \( f \in \mathcal{S}'(\mathbb{R}^n) \), \( G_\sigma(f) = F^{-1}(1 + |X|^2)^{\sigma/2}F(f) \). Then \( \| G_\sigma(f) \|_{B_p^{s-p,p}(\mathbb{R}^n)} \) is equivalent to the quasi-norm of \( B_p^{s,p}(\mathbb{R}^n) \).

The following lemma is easily verified by using the definition of \( d \)-set.

\textbf{Lemma 2.2} Let \( \Gamma \) be a \( d \)-set (0 < \( d < n \)). Then for \( \epsilon > 0, \quad 0 < r < 1, \) and \( p > \frac{d}{n} \)

\[ \int_{\Gamma} \frac{r^\epsilon}{(r + |X|)^{n+\epsilon}}d\mu(X) \sim r^{np-d}. \]

Denote by \( \text{Tr} \) the trace operator, initially defined on \( \mathcal{S}(\mathbb{R}^n) \) as the restriction to \( \Gamma \), since \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( B_p^{s,p}(\mathbb{R}^n) \). We also use \( E \) to denote the extension operator that extends functions from \( \Gamma \) to \( \mathbb{R}^n \).
Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1** For the theorem about $p \geq 1$, readers can refer to Theorem 1 in Chapter V in [2]. We only need to deal with the case $p < 1$.

Step 1. Firstly we should prove that

$$B^β,p_p(\Gamma) \subset B^α,p_p(\mathbb{R}^n)|_Γ.$$  

Since $B^β,p_p(\Gamma) \subset L^p(\Gamma)$, this is verified if

$$L_p(\Gamma) \subset B^α,p_p(\mathbb{R}^n)|_Γ.$$  

That means we need to prove for $f \in L_p(\Gamma)$, there exists $E$ such that

$$\|E(f)\|_{B^α,p_p(\mathbb{R}^n)} \leq C\|f\|_{L_p(\Gamma)}.$$  

In fact, such extension operator $E$ has been established in [5, p. 140]. We only need some little modifications of the corresponding part in [5] to fit our case. Thus we omit its proof here.

Step 2. Next we prove that

$$B^α,p_p(\mathbb{R}^n)|_Γ \subset B^β,p_p(\Gamma).$$  

With Lemma 2.1, it suffices to prove for $g \in S′(\mathbb{R}^n)$

$$\|\text{Tr}(G_{-α}(g))(\cdot)\|_{B^β,p_p(\Gamma)} \leq C\|g(\cdot)\|_{B^0,p_p(\mathbb{R}^n)}.$$  

Set $S(\cdot) \in S(\mathbb{R}^n)$ satisfying

$$\int S(X)dX \neq 0 \quad \text{and} \quad \hat{S}(\xi) \in D(\mathbb{R}^n),$$  

and supp$\hat{S}(\xi) \subset B(0,1)$. Set $S_k(X) = 2^{kn}S(2^kX)$, $D_k(X) = S_k - S_{k-1}$ and $D_0 = S$. We also assume that

$$\int D_k(X)X^βdX = 0(\beta \geq 0).$$  

Set

$$\tilde{G}_{-α} = \sum_{k=0}^{∞}2^{-kα}D_k.$$  

Using the Fourier analysis, we can easily obtain that when $|\xi| < 1$, $\tilde{G}_{-α}(\xi) \sim 1$; when $|\xi| \geq 1$, $\tilde{G}_{-α}(\xi) \sim |\xi|^{-α}$. Thus we can substitute the standard Bessel potential operator $G_{-α}$ with $\tilde{G}_{-α}$ in the process of proof. Next we use the so-called $ϕ$-transform to denote distribution function [6], for $g \in S′(\mathbb{P}(\mathbb{R}^n))$

$$g(\cdot) = \sum_{i(Q)=1}^{∞}\langle g, φ_Q \rangle \hat{φ}_Q + \sum_{m=1}^{∞}\sum_{i(Q)=2^{-m}}^{∞}\langle g, ψ_Q \rangle \hat{ψ}_Q,$$

where

$$Q = Q_{k,μ} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : μ_i \leq 2^μx_i < μ_i + 1, μ \in \mathbb{Z}^n, i = 1, \ldots, n\},$$

and $φ_Q, \hat{φ}_Q, ψ_Q, \hat{ψ}_Q \in S(\mathbb{R}^n)$ satisfy

$$|\partial_γφ_Q|, |\partial_γ\hat{φ}_Q|, |\partial_γψ_Q|, |\partial_γ\hat{ψ}_Q| \leq C_γL(\frac{|Q|^{\frac{γ}{2}}}{(1 + i(Q)^{-1}|X - X_Q|)^{L+γ}},$$
where $X_Q$ is the center of cube $Q$, $L, |\gamma| > 0$ and $\psi_Q, \tilde{\psi}_Q$ have vanishing moments up to some needed order, while $\phi_Q, \tilde{\phi}_Q$ may not have vanishing conditions. Now we need to deal with the term $\|\tilde{G}_{-\alpha}(g)\|_{L^p(\Gamma)}$.

$$
\|\tilde{G}_{-\alpha}(g)\|_{L^p(\Gamma)}^p = \int_{\Gamma} \left| \sum_{k=0}^{\infty} 2^{-ka} D_k \ast g(X) \right|^p d\mu(X)
$$

$$
\leq \int_{\Gamma} \left| \sum_{k=0}^{\infty} 2^{-ka} D_k \ast \left\{ \sum_{l(Q) = 1} \langle g, \phi_Q \rangle \phi_Q + \sum_{m=1}^{\infty} \sum_{l(Q) = 2^{-m}} \langle g, \psi_Q \rangle \psi_Q \right\} \right|^p d\mu(X).
$$

Assume that $g_1 = \sum_{l(Q) = 1} \langle g, \phi_Q \rangle \phi_Q$ and $g_2 = \sum_{m=1}^{\infty} \sum_{l(Q) = 2^{-m}} \langle g, \psi_Q \rangle \psi_Q$. Notice that for some $\epsilon > 0$ and $a = \min\{a, b\}$, $l(Q) \sim 2^{-m}$,

1. $|D_0 \ast \tilde{\phi}_Q(\cdot)| \leq C \frac{1}{(1 + |X - X_Q|)^{n+\epsilon}}$;
2. $|D_0 \ast \tilde{\psi}_Q(\cdot)| \leq C|Q|^{1/2-\epsilon/n} \frac{1}{(1 + |X - X_Q|)^{n+\epsilon}}$;
3. $|D_K \ast \tilde{\phi}_Q(\cdot)| \leq C 2^{-k\epsilon} \frac{1}{(1 + |X - X_Q|)^{n+\epsilon}}$, for $k > 0$;
4. $|D_K \ast \tilde{\psi}_Q(\cdot)| \leq C|Q|^{1/2-|m|\epsilon} \frac{2^{-(k+m)\epsilon}}{(2^{-(k+m)} + |X - X_Q|)^{n+\epsilon}}$, for $k > 0$.

Let us deal with the term $\int_{\Gamma} \sum_{k=0}^{\infty} 2^{-ka} D_k \ast g_2|^p d\mu(X)$. The other terms are similar and easy. By using $(a + b)^p \leq a^p + b^p (p \leq 1)$ and (2.4), we have

$$
\int_{\Gamma} \left| \sum_{k=0}^{\infty} 2^{-ka} D_k \ast g_2 \right|^p d\mu(X)
$$

$$
\leq C \int_{\Gamma} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \sum_{l(Q) = 2^{-m}} 2^{-k\alpha m + np |\gamma|/2 - |k-m|\epsilon} \frac{2^{-(k+m)\epsilon}}{(2^{-(k+m)} + |X - X_Q|)^{n+\epsilon}} \left| \langle g, \psi_Q \rangle \right|^p d\mu(X)
$$

$$
\leq C \left( \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \sum_{l(Q) = 2^{-m}} 2^{-k\alpha m + np |\gamma|/2 - |k-m|\epsilon} \left| \langle g, \psi_Q \rangle \right|^p + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \sum_{l(Q) = 2^{-m}} 2^{mp \alpha m + np |\gamma|/2 - |k-m|\epsilon} \left| \langle g, \psi_Q \rangle \right|^p \right).
$$

Note that $\alpha > \frac{n-d}{p}$ and $p > \frac{d}{n}$ and take $\epsilon > \frac{n}{p} - n$. Then we have

$$
\int_{\Gamma} \left| \sum_{k=0}^{\infty} 2^{-ka} D_k \ast g_2 \right|^p d\mu(X) \leq C \sum_{m=1}^{\infty} \sum_{l(Q) = 2^{-m}} 2^{np/(1 - 2/1/p)} \left| \langle g, \psi_Q \rangle \right|^p.
$$

Here we have just used Theorem 7.1 in [7]. It follows $\|\tilde{G}_{-\alpha}(g)\|_{L^p(\Gamma)}^p \leq C \|g\|_{B^p_{2^p/p}(\mathbb{R}^n)}^p$. Finally we prove

$$
\left( \int_{X,Y \in \Gamma} \left| \frac{|\tilde{G}_{-\alpha}(g)(X) - \tilde{G}_{-\alpha}(g)(Y)|^p}{|X - Y|^{d+2\beta}} d\mu(X) d\mu(Y) \right|^{1/p} \right)^{1/p} \leq C \|g\|_{B^p_{2^p/p}(\mathbb{R}^n)}.
$$
We have
\[
\int \int_{X,Y \in \Gamma} \frac{|\tilde{G}_{-\alpha}(g)(X) - \tilde{G}_{-\alpha}(g)(Y)|^p}{|X - Y|^{d + \beta p}} d\mu(X) d\mu(Y)
\]
\[
\leq \int \int_{X,Y \in \Gamma} \sum_{k=0}^{\infty} 2^{-k\alpha} \frac{|D_k(g)(X) - D_k(g)(Y)|^p}{|X - Y|^{d + \beta p}} d\mu(X) d\mu(Y)
\]
\[
\leq \sum_{k=0}^{\infty} 2^{-k\alpha} \int \int_{X,Y \in \Gamma} \frac{|\int_{\mathbb{R}^n} (D_k(X,Z) - D_k(Y,Z))g(Z) dZ|^p}{|X - Y|^{d + \beta p}} d\mu(X) d\mu(Y)
\]
\[
\leq \sum_{k=0}^{\infty} 2^{-k\alpha} \int \int_{\{X,Y \in \Gamma : |X - Y| \geq \frac{1}{2} 2^{-k}\}} + \int \int_{\{X,Y \in \Gamma : |X - Y| < \frac{1}{2} 2^{-k}\}} \ldots.
\]

Note that for the first term,
\[
\int_{\{X \in \Gamma : |X - Y| \geq \frac{1}{2} 2^{-k}\}} \frac{1}{|X - Y|^{d + \beta p}} d\mu(X) = \int_{\{Y \in \Gamma : |X - Y| \geq \frac{1}{2} 2^{-k}\}} \frac{1}{|X - Y|^{d + \beta p}} d\mu(Y) \sim C 2^{k\beta p},
\]
then the following step is almost the same as with the term \(\|\tilde{G}_{-\alpha}(g)\|_{L^p(\Gamma)}\). Now we deal with the second term. Notice that
\[
|D_k(X,Z) - D_k(Y,Z)| \leq C \frac{|X - Y|^\epsilon}{(2^{-k} + |X - Z|)^{\epsilon}} \left( \frac{2^{-k\epsilon}}{(2^{-k} + |X - Z|)^{n+\epsilon}} + \frac{2^{-k\epsilon}}{(2^{-k} + |Y - Z|)^{n+\epsilon}} \right),
\]
for \(|X - Y| \leq \frac{1}{2}(2^{-k} + |X - Z|)\).

Take \(\beta p < \epsilon = \beta p + \epsilon_0 < 1\), then we have
\[
\int_{\{X \in \Gamma : |X - Y| < \frac{1}{2} 2^{-k}\}} \frac{1}{|X - Y|^{d - \epsilon_0}} 2^{k\beta p + k\epsilon_0} d\mu(X)
\]
\[
= \int_{\{Y \in \Gamma : |X - Y| < \frac{1}{2} 2^{-k}\}} \frac{1}{|X - Y|^{d - \epsilon_0}} 2^{k\beta p + k\epsilon_0} d\mu(Y) \sim C 2^{k\beta p}.
\]
Then we can obtain
\[
\int \int_{X,Y \in \Gamma} \frac{|\tilde{G}_{-\alpha}(g)(X) - \tilde{G}_{-\alpha}(g)(Y)|^p}{|X - Y|^{d + \beta p}} d\mu(X) d\mu(Y) \leq C \|g\|_{B^{\alpha,p}_p(\mathbb{R}^n)}.
\]
Thus we have proved \(B^{\alpha,p}_p(\mathbb{R}^n) \mid \Gamma \subset B^{\beta,p}_p(\Gamma)\). The proof is completed. \(\square\)

**Remark 2.1** Notice that in our setting in the main theorem, \(\Gamma\) is a \(d\)-set with radius \(r_0 > 0\). In fact from the proof of our main theorem, one finds that \(r_0\) can be infinite. In this case we need the moment conditions on the kernel of \(D_k\) \((k > 0)\). That means \(p > d/n\) which connects with the moment conditions of the kernel \(D_k\). If \(r_0 < \infty\), \(\Gamma\) is a compact set when \(0 < d < n\), which means \(\Gamma\) is porous (see related definition in [9] or [10]). In that case, related results can be found in [8] where similar trace theorem holds in the case \(0 < p < \infty\). Hence this is the difference between our results and the works of Triebel [8–10].

**References**


