On Generalized $E$-Monotonicity in Banach Spaces

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Abstract
Several new kinds of generalized $E$-preinvexity and generalized invariant $E$-monotonicity are introduced in the setting of Banach spaces. The relations between $E$-preinvexity, $E$-prequasiinvexity, (pseudo, quasi) $E$-invexity and invariant (pseudo, quasi) $E$-monotonicity are studied, which can be viewed as an extension of some known results.

Keywords $E$-preinvexity; $E$-invexity; invariant $E$-monotonicity; relations.

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1. Introduction

Convexity is a common assumption made in mathematical programming. There have been increasing attempts to weaken the convexity of objective functions, see for example [1–9] and references therein. An interesting generalization for convexity is $E$-convexity, which was introduced and studied by Youness [1–3] and Yang [4]. They studied characterizations of efficient solutions and optimality criteria for a class of $E$-convex programming problems. Later, $E$-quasiconvexity was introduced in [5] and some basic properties for $E$-convex and $E$-quasiconvex functions were developed there. Very recently, Fulga and Preda [6] introduced $E$-invex sets and $E$-preinvex and $E$-prequasiinvex functions as an extension of the $E$-convex sets and $E$-convex and $E$-quasiconvex functions, respectively.

A concept related to the convexity is the monotonicity of mappings. In 1990, Karamardian and Schaible [7] studied the relations between the convexity of a real-valued function and the monotonicity of its gradient mapping. Yang et al. [8] investigated the relations between invexity and generalized invariant monotonicity in $R^n$. In this paper, we will introduce several new notions of generalized invexity and generalized invariant monotonicity, which are called $E$-quasinvexity, $E$-pseudoinvexity and invariant pseudo (quasi) $E$-monotonicity, and study their relations in Banach spaces. The results here can be viewed as an extension and improvement of corresponding results in [7, 8].

2. Invariant $E$-monotonicity

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Throughout this paper, let \( X \) be a real Banach space and \( M \) a nonempty subset of \( X \). Let \( \eta : \times X \to X \) and \( E : X \to X \) be single-valued mappings and \( f : X \to R \) a function.

The set \( M \) is said to be in\( x \) with respect to \( \eta \) (see [9]) if for any \( x, y \in M \) and any \( \lambda \in [0,1] \), one has \( x + \lambda \eta(y,x) \in M \), and \( E \)-in\( x \) with respect to \( \eta \) (see [6]) if for any \( x, y \in M \) and any \( \lambda \in [0,1] \) one has \( E(x) + \lambda \eta(E(y), E(x)) \in M \). For the sake of brevity, \( E(x) \) will be written as \( Ex(x) \) for any \( x \in M \).

The function \( f \) is said to be Gâteaux differentiable at \( x_0 \in M \) if there exists a linear function \( f'(x_0) : X \to R \) such that

\[
\langle f'(x_0), v \rangle = \lim_{t \downarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}, \quad \forall v \in X,
\]

where \( f'(x_0) \) is called the Gâteaux derivative of \( f \) at \( x_0 \). The function \( f \) is said to be Gâteaux differentiable on \( M \) if it is Gâteaux differentiable at every \( x \) in \( M \).

**Definition 2.1** Let \( M \) be an \( E \)-in\( x \) set with respect to \( \eta \) and \( f : X \to R \) be a Gâteaux differentiable function on \( M \). The function \( f \) is said to be

(i) ([6]) \( E \)-prein\( x \) in \( M \) with respect to \( \eta \) if

\[
f(Ey + \lambda \eta(Ex, Ey)) \leq \lambda f(Ex) + (1 - \lambda)f(Ey), \quad \forall x, y \in M, \forall \lambda \in [0,1];
\]

(ii) \( E \)-in\( x \) on \( M \) with respect to \( \eta \) if

\[
\langle f'(Ey), \eta(Ex, Ey) \rangle \leq f(Ex) - f(Ey), \quad \forall x, y \in M.
\]

**Definition 2.2** Let \( M \) be an \( E \)-in\( x \) set with respect to \( \eta \) and \( f : X \to R \) be a Gâteaux differentiable function on \( M \). The operator \( f' \) is said to be invariant \( E \)-monotone on \( M \) with respect to \( \eta \) if

\[
\langle f'(Ey), \eta(Ex, Ey) \rangle + \langle f'(Ex), \eta(Ey, Ex) \rangle \leq 0, \quad \forall x, y \in M.
\]

When \( E = I \), the identity mapping, Definitions 2.1 and 2.2 reduce to the concepts of prein\( x \) vexity, in\( x \) vexity and invariant mon\( x \) tonicity in [8], respectively.

Motivated by the previous works on this issue, in this paper, we will study the closed relations between \( E \)-prein\( x \) vexity, \( E \)-in\( x \) vexity and invariant \( E \)-monotonicity. For this end, we first recall the following two assumptions, which are taken from [8, 9] and used in many papers. Let \( K \) be a nonempty subset of \( X \).

**Assumption A** Let the set \( K \) be in\( x \) with respect to \( \eta \) and \( f : K \to R \) be a function. Assume that \( f(y + \eta(x,y)) \leq f(x) \) for all \( x, y \in K \).

**Assumption C** Let \( \eta : X \times X \to X \) be a mapping. Assume that for any \( x, y \in K \) and any \( \lambda \in [0,1] \) one has

\[
\eta(y, y + \lambda \eta(x,y)) = -\lambda \eta(x,y) \quad \text{and} \quad \eta(x, y + \lambda \eta(x,y)) = (1 - \lambda) \eta(x,y).
\]

Yang et al. [8] showed that if \( \eta \) satisfies Assumption C, then

\[
\eta(y + \lambda_1 \eta(x,y), y + \lambda_2 \eta(x,y)) = (\lambda_1 - \lambda_2) \eta(x,y), \quad \forall x, y \in K, \lambda_1, \lambda_2 \in [0,1].
\]
The following example shows that the converse implication is not necessarily true, that is, the equality (1) is a proper generalization of Assumption C’.

**Example 2.1** Let $K = [-2, 2]$ and

$$
\eta(x, y) = \begin{cases} 
  x - y, & x y \geq 0, \\
  -\frac{y}{2}, & x > 0, y < 0, \\
  2 - y, & x < 0, y > 0.
\end{cases}
$$

We can verify that $\eta$ satisfies the equality (1). But Assumption C’ does not hold for $x > 0, y < 0$ and $\lambda \in (0, 1]$.

Now, we introduce two similar assumptions, which are used in the sequel.

**Assumption A** Let $f : X \to R$ be a function and $\eta : X \times X \to X$ and $E : X \to X$ be single-valued mappings. Assume that $f(Ey + \eta(EX, Ey)) \leq f(EX), \forall x, y \in M$.

**Assumption C** Let $\eta : X \times X \to X$ and $E : X \to X$ be two single-valued mappings. Assume that for any $x, y \in M$ and any $\lambda_1, \lambda_2 \in [0, 1]$ one has

$$
\eta(Ey + \lambda_1 \eta(EX, Ey), Ey + \lambda_2 \eta(EX, Ey)) = (\lambda_1 - \lambda_2)\eta(EX, Ey).
$$

**Theorem 2.1** Let $M$ be an $E$-invex set with respect to $\eta$, $E(M)$ an invex set with respect to $\eta$ and $f : X \to R$ Gâteaux differential on $M$. Then

(i) $E$-preinvexity of $f$ implies $E$-invexity of $f$ on $M$ with respect to $\eta$;

(ii) $E$-invexity of $f$ implies invariant $E$-monotonicity of $f'$ on $M$ with respect to $\eta$;

(iii) If Assumptions A and C are both satisfied, then invariant $E$-monotonicity of $f'$ implies $E$-preinvexity of $f$ on $M$ with respect to $\eta$.

**Proof**

(a) Let $f$ be $E$-preinvex on $M$. Then for any $x, y \in M$ and any $\lambda \in (0, 1]$ one has

$$
\frac{f(Ey + \lambda\eta(EX, Ey)) - f(Ey)}{\lambda} \leq f(EX) - f(Ey).
$$

Taking the limit for the above inequality as $\lambda \downarrow 0$, we get

$$
(f'(Ey), \eta(EX, Ey)) \leq f(EX) - f(Ey),
$$

which shows that $f$ is $E$-invex on $M$.

(b) Let $f$ be $E$-invex on $M$. Then the inequality (2) holds for any $x, y \in M$ and then

$$
(f'(Ey), \eta(EX, Ey)) + (f'(EX), \eta(EY, EX)) \leq 0,
$$

which indicates that $f'$ is invariant $E$-monotone on $M$.

(c) Let $f'$ be invariant $E$-monotone on $M$ and Assumptions A and C hold. Assume to the contrary that $f$ is not $E$-preinvex on $M$. Then there exist $x^0, y^0 \in M$ and $\lambda^0 \in (0, 1)$ such that

$$
f(EX^0) > \lambda^0 f(EX^0) + (1 - \lambda^0) f(Ey^0),
$$

where $EX^0 = Ey^0 + \lambda^0 \eta(EX^0, Ey^0)$. By Assumption A, we have $f(EX^0) > \lambda^0 f(Ey^0 + \eta(EX^0, Ey^0)) + (1 - \lambda^0) f(Ey^0)$, that is,

$$
\lambda^0 (f(EX^0) - f(Ey^0 + \eta(EX^0, Ey^0))) + (1 - \lambda^0) (f(EX^0) - f(Ey^0)) > 0.
$$
By the mean-value theorem, we get
\[ (f'(Ez^1), \lambda^0(\lambda^0 - 1)\eta(Ez^0, Ey^0)) + (f'(Ez^2), (1 - \lambda^0)\lambda^0\eta(Ez^0, Ey^0)) > 0, \]
where \( Ez^1 = Ey^0 + \lambda^1\eta(Ez^0, Ey^0) \), \( Ez^2 = Ey^0 + \lambda^2\eta(Ez^0, Ey^0) \) and \( 0 < \lambda^2 < \lambda^0 < \lambda^1 < 1 \). By Assumption C, it follows from (3) that
\[ (f'(Ez^1), \eta(Ez^2, Ez^1)) + (f'(Ez^2), \eta(Ez^1, Ez^2)) > 0, \]
which contradicts the invariant \( E \)-monotonicity of \( f' \). Therefore, the assertion (iii) holds. \( \square \)

3. Generalized \( E \)-invexity and generalized \( E \)-monotonicity

In this section, we will introduce several new kinds of generalized \( E \)-invexity and generalized \( E \)-monotonicity, and establish the relationships between generalized \( E \)-invexity of the function \( f \) and generalized \( E \)-monotonicity of its Gâteaux differential \( f' \).

**Definition 3.1** Let \( M \) be an \( E \)-invex set with respect to \( \eta \). A Gâteaux differential function \( f : X \rightarrow R \) is said to be

(i) \([6]\) \( E \)-prequasiinvex on \( M \) with respect to \( \eta \) if
\[ f(Ey + \lambda\eta(Ex, Ey)) \leq \max\{f(Ex), f(Ey)\}, \quad \forall x, y \in M, \forall \lambda \in [0, 1]; \]

(ii) \( E \)-quasiinvex on \( M \) with respect to \( \eta \) if
\[ f(Ex) \leq f(Ey) \Rightarrow (f'(Ey), \eta(Ex, Ey)) \leq 0, \quad \forall x, y \in M; \]

(iii) \( E \)-pseudoinvex on \( M \) with respect to \( \eta \) if
\[ (f'(Ey), \eta(Ex, Ey)) \geq 0 \Rightarrow f(Ex) \geq f(Ey), \quad \forall x, y \in M. \]

**Definition 3.2** Let \( M \) be an \( E \)-invex set with respect to \( \eta \) and \( f : X \rightarrow R \) a Gâteaux differential function. The operator \( f' \) is said to be

(i) invariant quasi \( E \)-monotone on \( M \) with respect to \( \eta \) if
\[ (f'(Ey), \eta(Ex, Ey)) > 0 \Rightarrow (f'(Ex), \eta(Ey, Ex)) \leq 0, \quad \forall x, y \in M; \]

(ii) invariant pseudo \( E \)-monotone on \( M \) with respect to \( \eta \) if
\[ (f'(Ey), \eta(Ex, Ey)) \geq 0 \Rightarrow (f'(Ex), \eta(Ey, Ex)) \leq 0, \quad \forall x, y \in M. \]

In the following, we study the relations between \( E \)-prequasiinvexity, (pseudo) quasi \( E \)-invexity and invariant (pseudo) quasi \( E \)-monotonicity.

**Theorem 3.1** Let \( M \) be an \( E \)-invex set with respect to \( \eta \), \( E(M) \) an invex set with respect to \( \eta \) and \( f : X \rightarrow R \) a Gâteaux differential function. Then

(i) \( E \)-prequasiinvexity of \( f \) implies \( E \)-quasiinvexity of \( f \) on \( M \) with respect to \( \eta \);

(ii) \( E \)-quasiinvexity of \( f \) implies invariant quasi \( E \)-monotonicity of \( f' \) on \( M \) with respect to \( \eta \);

(iii) If Assumptions A and C are satisfied, then invariant quasi \( E \)-monotonicity of \( f' \) implies \( E \)-prequasiinvexity of \( f \) on \( M \) with respect to \( \eta \).
Proof (a) Let $f$ be $E$-prequasiinvariant on $M$. For any $x, y \in M$ and any $\lambda \in (0, 1)$, assume without loss of generality that $f(Ey) \leq f(Ey)$. Then $f(Ey + \lambda \eta(Ey, Ey)) \leq f(Ey)$, that is, $\langle f(Ey + \lambda \eta(Ey, Ey)) - f(Ey), Ey \rangle \leq 0$. Taking the limit for the last inequality as $\lambda \downarrow 0$, we get $\langle f'(Ey), \eta(Ey, Ey) \rangle \leq 0$, which indicates that $f$ is $E$-quasiinvariant on $M$.

(b) Let $f$ be $E$-quasiinvariant on $M$. Assume to the contrary that $f'$ is not invariant quasi $E$-monotone on $M$. Then there exist $x^0, y^0 \in M$ such that

$$\langle f'(Ey^0), \eta(Ey^0, Ey^0) \rangle > 0 \Rightarrow \langle f'(Ey^0), \eta(Ey^0, Ey^0) \rangle > 0.$$

By the definition of $E$-quasiinvariant, we get $f(Ey^0) > f(Ey^0)$ and $f(Ey^0) > f(Ey^0)$, which is a contradiction. Therefore, the assertion (ii) holds.

(c) Let $f'$ be invariant quasi $E$-monotone on $M$ and Assumptions A and C hold. Assume to the contrary that $f$ is not $E$-quasiinvariant. Then there exist $x^0, y^0 \in M$ and $\lambda^0 \in (0, 1)$ such that

$$f(Ey^0 + \lambda^0 \eta(Ey^0, Ey^0)) > \max\{f(Ey^0), f(Ey^0)\}.$$

By the mean-value theorem and Assumption A, there exist $\lambda^1 \in (0, \lambda^0)$ and $\lambda^2 \in (\lambda^0, 1)$ such that

$$0 < f(Ez^0) - f(Ey^0) = \langle f'(Ez^1), \lambda^0 \eta(Ey^0, Ey^0) \rangle$$

and

$$0 < f(Ez^0) - f(Ey^0 + \eta(Ey^0, Ey^0)) = \langle f'(Ez^2), (\lambda^0 - 1) \eta(Ey^0, Ey^0) \rangle,$$

where $Ez^0 = Ey^0 + \lambda^0 \eta(Ey^0, Ey^0)$, $Ez^1 = Ey^0 + \lambda^1 \eta(Ey^0, Ey^0)$ and $Ez^2 = Ey^0 + \lambda^2 \eta(Ey^0, Ey^0)$. From Assumption C, it follows that $\langle f'(Ez^1), \eta(Ez^2, Ez^1) \rangle > 0$ and $\langle f'(Ez^2), \eta(Ez^1, Ez^2) \rangle > 0$ which is a contradiction to the invariant quasi $E$-monotonicity of $f'$. So the assertion (iii) holds. □

Theorem 3.2 Let $M$ be an $E$-invex set with respect to $\eta$, $E(M)$ an invex set with respect to $\eta$ and $f : X \to R$ a Gâteaux differential function. If Assumption A holds and $f$ is $E$-pseudoinvariant on $M$ with respect to $\eta$, then $f$ is $E$-prequasiinvariant on $M$ with respect to $\eta$.

Proof Assume to the contrary that $f$ is not $E$-prequasiinvariant with respect to $\eta$. Then there exist $x^0, y^0 \in M$ and $\lambda^0 \in (0, 1)$ such that

$$f(Ey^0 + \lambda^0 \eta(Ey^0, Ey^0)) > \max\{f(Ey^0), f(Ey^0)\}.$$

Suppose without loss of generality that $f(Ey^0) \leq f(Ey^0)$. By Assumption A, we have

$$f(Ey^0 + \lambda^0 \eta(Ey^0, Ey^0)) \geq f(Ey^0) \leq f(Ey^0 + \eta(Ey^0, Ey^0)),$$

which indicates that there exists $\lambda^0 \in (0, 1)$ such that

$$f(Ey^0 + \lambda^0 \eta(Ey^0, Ey^0)) > f(Ey^0),$$

where $Ey^0 = Ey^0 + \lambda^0 \eta(Ey^0, Ey^0)$. Consequently, $f'(Ey^0) = 0$ and then $\langle f'(Ey^0), \eta(Ey^0, Ey^0) \rangle = 0$. It follows from the pseudoinvexity of $f$ that $f(Ey^0) \geq f(Ey^0)$, which is a contradiction. Thus, the assertion of the theorem is true.

Theorem 3.3 Let $M$ be an $E$-invex set with respect to $\eta$, $E(M)$ an invex set with respect
to \( \eta \) and \( f : X \to R \) a Gâteaux differential function. If Assumptions A and C hold, then \( f \) is E-pseudoinvex on \( M \) with respect to \( \eta \) if and only if \( f' \) is invariant pseudo E-monotone on \( M \) with respect to \( \eta \).

**Proof** Let \( f \) be E-pseudoinvex on \( M \) with respect to \( \eta \). Assume to the contrary that \( f' \) is not invariant pseudo E-monotone on \( M \) with respect to \( \eta \). Then there exist \( x^0, y^0 \in M \) such that

\[
\langle f'(Ey^0), \eta(Ex^0, Ey^0) \rangle \geq 0 \Rightarrow \langle f'(Ex^0), \eta(Ey^0, Ex^0) \rangle > 0.
\]  

(4)

By the E-pseudoinvexity of \( f \), we have

\[
\langle f'(Ey^0), \eta(Ex^0, Ey^0) \rangle \geq 0 \Rightarrow f(Ex^0) \geq f(Ey^0).
\]  

(5)

According to Theorems 3.2 and 3.1(i), it follows from (5) that \( \langle f'(Ex^0), \eta(Ey^0, Ex^0) \rangle \leq 0 \), which contradicts the implication (4).

Conversely, let \( f' \) be invariant pseudo E-monotone on \( M \) with respect to \( \eta \). For any \( x, y \in M \) with

\[
\langle f'(Ey), \eta(Ex, Ey) \rangle \geq 0,
\]  

(6)

we want to show that \( f(Ex) \geq f(Ey) \). Assume to the contrary that \( f(Ex) < f(Ey) \). By the mean-value theorem, there exists \( \lambda^0 \in (0, 1) \) such that

\[
f(Ey + \lambda^0 \eta(Ex, Ey)) - f(Ey) = \langle f'(Ey + \lambda^0 \eta(Ex, Ey)), \eta(Ex, Ey) \rangle.
\]

By Assumptions A and C, it follows that

\[
\langle f'(Ey + \lambda^0 \eta(Ex, Ey)), \eta(Ey, Ey + \lambda^0 \eta(Ex, Ey)) \rangle > 0.
\]

By the invariant pseudo E-monotonicity of \( f' \), we can deduce that \( \langle f'(Ey), \eta(Ex, Ey) \rangle < 0 \), which contradicts the inequality (6). Hence, \( f \) is E-pseudoinvex on \( M \) with respect to \( \eta \). □

**References**


