A Stage-Structured Predator-Prey System with Impulsive Effect and Holling Type-II Functional Response

Ling Shu WANG\textsuperscript{1,∗}, Rui XU\textsuperscript{2}, Guang Hui FENG\textsuperscript{2}

1. School of Mathematics and Statistics, Hebei University of Economics & Business, Shijiazhuang 050061, P. R. China;
2. Institute of Applied Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, P. R. China

Abstract A stage-structured predator-prey system with impulsive effect and Holling type-II functional response is investigated. By the Floquet theory and small amplitude perturbation skills, it is proved that there exists a global stable pest-eradication periodic solution when the impulsive period is less than some critical values. Further, the conditions for the permanence of system are established. Numerical simulations are carried out to illustrate the impulsive effect on the dynamics of the system.

Keywords predator-prey system; Holling type-II functional response; impulsive effect; extinction; permanence.

1. Introduction

Controlling insects and other arthropods has become an increasing complex issue over the past two decades. Minimizing losses due to insect pests and insect vectors of important plant, animal and human diseases remains an essential component of the programs in the Office of Agriculture Entomology. With the development of society and progress of science and technology, man has adopted some advanced and modern weapons, for instance chemical pesticides, biological pesticides. Some brilliant achievements have been obtained. A great deal of and a large variety of pesticides are used to control pests. In all, pesticides are useful because they can quickly kill a significant portion of a pest population and sometimes provide the only feasible method for preventing economic loss. However, pesticides pollution is also recognized as a major health hazard to human beings and beneficial insects.

Biological control is the practice of using natural enemies such as predators and parasites to suppress a pest population [1, 3, 6]. This method is typically used on the control of pests that have been imported from foreign regions. Virtually, all insects and mite pests have some natural enemies. These natural enemies can help prevent insect pest populations from increasing...
to levels that cause economic damage. This can be achieved by mass production and periodic release of natural enemies of the pest, and by genetic enhancement of the enemies to increase their effectiveness at control. The pioneering project of biological control began in 1888 when the new legendary predator, the Vedalia beetle, was imported from Australia and established in California, where it rapidly suppressed populations of cottoning cushion scale that had been decimating the developing citrus industry [2]. Further, the biological control method is harmless to man, animals and the environments.

Stage-structured models have received great attention in recent years. Two species population models with stage structure were discussed by several authors (see, for example [7–9]). In [7], Wang and Chen proposed a predator-prey system with stage structure under the assumptions that the predators are divided into two groups, one immature and the other mature, and that only mature predators can attack prey and have reproductive ability, while immature predators do not attack prey and have no reproductive ability.

Motivated by the work developed in Wang [7], in this paper, we are concerned with the impulsive effect on the dynamics of the following delayed differential equations

\[
\begin{align*}
&\dot{x}(t) = rx(t) - ax^2(t) - \frac{a_1x(t)y_2(t)}{1 + mx(t)} \\
&\dot{y}_1(t) = \frac{a_2x(t)y_2(t)}{1 + mx(t)} - (r_1 + d)y_1(t) \\
&\dot{y}_2(t) = dy_1(t) - r_2y_2(t) \\
&\Delta x(t) = 0 \\
&\Delta y_1(t) = p_1(t) \\
&\Delta y_2(t) = p_2(t)
\end{align*}
\]

where \(x(t)\) is the density of the prey population at time \(t\), \(y_1(t)\) and \(y_2(t)\) are the densities of the immature and mature predators at time \(t \neq n\tau\), respectively. \(\Delta y_i(t) = y_i(t) - y_i(t^-)\). The meanings of positive parameters \(a, a_1, a_2, d, r, r_1\) and \(r_2\) can be seen in [8]. \(p_1\) and \(p_2\) are the release amount of the immature and mature predators at time \(t = n\tau\), respectively. \(\tau\) is the period of the impulsive effect.

The organization of this paper is as follows. In the next section, we give some notations and lemmas. In Section 3, by the Floquet theory and small amplitude perturbation skills, we prove that there exists a global stable pest-eradication periodic solution when the impulsive period is less than some critical values. In Section 4, we derive the condition for the permanence of the system (1) via comparison theorem of impulsive differential equations. Numerical simulations are carried out in Section 5 to illustrate the dynamics of the system including period-doubling bifurcation, quasi-doubling oscillation and chaos.

2. Preliminaries

In this section, we will give some definitions and Lemmas which will be useful for our main results. Let \(R^+ = [0, +\infty), V : R^+ \times R^3_+ \rightarrow R^+_+\). Then \(V\) is said to belong to class \(V_0\) if

(i) \(V\) is continuous in \(((n-1)\tau, n\tau] \times R^3_+\) and for each \(z \in R^3_+, n \in Z^+_+, \lim_{(t,y) \rightarrow (n\tau+, z)} V(t, y) =

Lemma 1
The following lemma is obvious.

Taking $0 < k \leq \min\{r_1, r_2\}$, we can obtain

$$D^+V(t) + kV(t) \leq \frac{a_2(r+k)^2}{4aa_1} = K, \quad t \neq n\tau;$$

$$V(n\tau) = V(n\tau^-) + p_1 + p_2, \quad t = n\tau.$$

By comparison theorem, we have

$$V(t) \leq V(0)e^{-kt} + \frac{K}{k}(1 - e^{-kt}) + \frac{(p_1 + p_2)(e^{-k(t-\tau)} - e^{k\tau})}{1 - e^{k\tau}} \quad \text{as } t \to \infty.$$

Therefore, $V(t)$ is ultimately bounded by a constant and there exists a constant $M > 0$, such that $x(t) \leq M, y_i(t) \leq M, i = 1, 2$, for each solution $z(t)$ of system (1) with $t$ large enough. □

3. Extinction

In this section, we study the stability of the pest-eradication periodic solution. We demonstrate below the existence and global stability of the solution corresponding to the extinction of the prey population, i.e., $x(t) = 0, t \geq 0$. Under this condition, the growth of predator
Lemma 3 System (2) has a globally asymptotically stable positive periodic solution $y^*(t) = (y_1^*(t), y_2^*(t))$, $n\tau < t \leq (n+1)\tau$, where

$$
y_1^*(t) = \frac{p_1 e^{-(r_1+d)(t-n\tau)}}{1-e^{-(r_1+d)\tau}},
y_2^*(t) = \frac{dp_1}{r_1+d-r_2} e^{-r_2(t-n\tau)} - \frac{dp_1 e^{-(r_1+d)(t-n\tau)}}{(r_1+d-r_2)(1-e^{-(r_1+d)\tau})}.
$$

Proof Integrating (2) between pulses $n\tau < t \leq (n+1)\tau$, we obtain

$$
y_1(t) = y_1(n\tau)e^{-(r_1+d)(t-n\tau)},
y_2(t) = y_2(n\tau)e^{-r_2(t-n\tau)} + \frac{dy_1(n\tau)}{r_1+d-r_2}(e^{-r_2(t-n\tau)} - e^{-(r_1+d)(t-n\tau)}),
y_1(n\tau) = p_1 + y_1(n\tau^-),
y_2(n\tau) = p_2 + y_2(n\tau^-).
$$

Thus, we deduce the stroboscopic map

$$
y_1((n+1)\tau) = p_1 + y_1(n\tau)e^{-(r_1+d)\tau},
y_2((n+1)\tau) = p_2 + y_2(n\tau)e^{-r_2\tau} + \frac{dy_1(n\tau)}{r_1+d-r_2}(e^{-r_2\tau} - e^{-(r_1+d)\tau}).
$$

The map has the unique positive fixed point

$$
\left(\frac{p_1}{1-e^{-(r_1+d)\tau}}, \frac{p_2}{1-e^{-r_2\tau}} + \frac{dp_1(e^{-r_2\tau} - e^{-(r_1+d)\tau})}{(r_1+d-r_2)(1-e^{-r_2\tau})(1-e^{-(r_1+d)\tau})}\right) = (y_1^*, y_2^*).
$$

The fixed point implies that there is a corresponding cycle of period $\tau$, i.e., when $n\tau < t \leq (n+1)\tau$,

$$
y_1(n\tau) = \frac{p_1 e^{-(r_1+d)(n\tau)}}{1-e^{-(r_1+d)\tau}},
y_2(n\tau) = \frac{dp_1}{r_1+d-r_2} e^{-r_2(n\tau)} - \frac{dp_1 e^{-(r_1+d)(n\tau)}}{(r_1+d-r_2)(1-e^{-(r_1+d)\tau})}.
$$

Further, using iteration (3) step by step, we have

$$
y_1(n\tau) = \frac{p_1(1 + e^{-(r_1+d)\tau} + e^{-2(r_1+d)\tau} + \cdots + e^{-(r_1+d)(n-1)\tau}) + y_1(0)e^{-(r_1+d)n\tau}}{1-e^{-(r_1+d)\tau}},
y_2(n\tau) = \frac{dp_1}{r_1+d-r_2}(e^{-r_2\tau} - e^{-(r_1+d)\tau}) + \frac{y_2(0)e^{-r_2n\tau}}{1-e^{-r_2\tau}} + y_2(0)e^{-r_2n\tau}.
$$
From this, we can obtain
\[
\begin{align*}
\lim_{n \to \infty} y_1(n\tau) &= \frac{p_1}{1 - e^{-(r_1+d)\tau}} = y_1^*, \\
\lim_{n \to \infty} y_2(n\tau) &= \frac{p_2}{1 - e^{-r_2\tau}} + \frac{dp_1(e^{-r_2\tau} - e^{-(r_1+d)\tau})}{(r_1 + d - r_2)(1 - e^{-r_2\tau})(1 - e^{-(r_1+d)\tau})} = y_2^*.
\end{align*}
\]
Thus, \( y^* = (y_1^*, y_2^*) \) is globally asymptotically stable, i.e., the periodic solution \( y^*(t) = (y_1^*(t), y_2^*(t)) \) is globally asymptotically stable. \( \square \)

From Lemma 3, we know system (1) has a periodic solution \((0, y_1^*(t), y_2^*(t))\). Now we show the global stability of the periodic solution.

Let \( x_1(t) = y_1(t) - y_1^*(t), \) \( x_2(t) = y_2(t) - y_2^*(t) \). Then linearized equations corresponding to system (1) read
\[
\begin{align*}
\dot{x}_1(t) &= (r - a_1y_2^*)x_1(t), \\
\dot{x}_2(t) &= a_2y_2^*x_1(t) - (r_1 + d)x_2(t), \\
\dot{x}_3(t) &= dx_2(t) - r_2x_3(t), \\
x_1(n\tau) &= x_1(n\tau^-), \\
x_2(n\tau) &= x_2(n\tau^-), \\
x_3(n\tau) &= x_3(n\tau^-),
\end{align*}
\]

Denote
\[
A(t) = \begin{pmatrix} r - a_1y_2^* & 0 & 0 \\ a_2y_2^* & -(r_1 + d) & 0 \\ 0 & d & -r_2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
The monodromy matrix \( M(T) \) of system (4) is
\[
M(\tau) = B \exp^{\int_0^\tau A(t)dt} = \begin{pmatrix} \exp^{\int_0^\tau (r - a_1y_2^*)dt} & 0 & 0 \\ * & \exp^{-(r_1+d)\tau} & 0 \\ 0 & \Delta & \exp^{-r_2\tau} \end{pmatrix}.
\]
There is no need to calculate the exact forms of * and \( \Delta \) as they are not required in analysis that follows. The Floquet multipliers are defined as the eigenvalues of the matrix \( M(\tau) \), i.e.,
\( \lambda_1 = \exp^{\int_0^\tau (r - a_1y_2^*)dt}, \lambda_2 = \exp^{-(r_1+d)\tau}, \lambda_3 = \exp^{-r_2\tau} \). It is obvious that \( \lambda_2 < 1 \) and \( \lambda_3 < 1 \), so the stability of \((0, y_1^*(t), y_2^*(t))\) is decided by \( \lambda_1 < 1 \). Hence, the periodic solution \((0, y_1^*(t), y_2^*(t))\) is locally stable provided \( \int_0^\tau (r - a_1y_2^*)dt = \tau r - \frac{a_1}{r_2} (p_2 + \frac{dp_1}{r_1 + d}) < 0. \)

**Theorem 1** Assume \((H_1)\) \( \tau < \frac{aa_1}{rr_2(a + mr)} \left( \frac{p_2}{p_1} + \frac{dp_1}{r_1 + d} \right) \) holds, then the periodic solution \((0, y_1^*(t), y_2^*(t))\) is globally asymptotically stable.

**Proof** In the following we prove the periodic solution is global attractivity. By the comparison theorem of impulsive differential equations and Lemma 3, we know that for any solution \( z(t) = (x(t), y_1(t), y_2(t)) \) of system (1) with initial value \( x(0) > 0, y_1(0) > 0, y_2(0) > 0 \), when \( t \) is large enough, the following inequality holds:
\[
y_2(t) > y_2^*(t) - \varepsilon.
\]
Further, from the first equation of system (1), we know (5) implies
\[ \dot{x}(t) < x(t)[r - b(y^*_2(t) - \varepsilon)] \]
where \( b = a\alpha_1/(a + nr) \), then we consider the following comparison system:
\[
\begin{aligned}
\dot{y}(t) &= y(t)(r - b(y^*_2(t) - \varepsilon)), \quad t \neq n\tau, \\
\Delta y(t) &= 0, \quad t = n\tau.
\end{aligned}
\]

Integrating system (6) between pulses \((n\tau, (n + 1)\tau)\), we have
\[ y((n + 1)\tau) = y(n\tau)e^{\int_{n\tau}^{(n + 1)\tau}(r - b(y^*_2(t) - \varepsilon))dt} = y(n\tau)e^{\int_{n\tau}^{(n + 1)\tau}(r - b(y^*_2(t) - \varepsilon))dt} = y(0)e^{n(\sigma + b\varepsilon\tau)}, \]
where \( \sigma = r\tau - b(p_2 + \frac{d\alpha_1}{r_1 + d}). \) By \((H_1)\), we can choose \( \varepsilon > 0 \) sufficiently small such that \( \sigma + b\varepsilon\tau < 0. \) Thus, we can obtain \( \lim_{n\to\infty} y(n\tau) = 0. \) On the other hand, integrating and solving the first equation of system (6) between pulses gives
\[ y(t) = y(n\tau)e^{\int_{n\tau}^{t}(r - b(y^*_2(t) - \varepsilon))dt}, \quad n\tau < t < (n + 1)\tau. \]
Incorporating the boundedness of \( e^{\int_{n\tau}^{t}(r - b(y^*_2(t) - \varepsilon))dt} \), we have \( \lim_{t\to\infty} y(t) = 0. \) According to the comparison theorem, we have \( \lim_{t\to\infty} \sup x(t) \leq \lim_{t\to\infty} \sup y(t) = 0. \) Considering the positivity of \( x(t) \), we know \( \lim_{t\to\infty} x(t) = 0. \)

When \( t \) is large enough, we can obtain \( 0 < x(t) < \varepsilon. \) The second equation of (1) implies
\[ \dot{y}_1(t) < a_2M\varepsilon - (r_1 + d)y_1(t). \]
Consider the following comparison system
\[
\begin{aligned}
\dot{y}_1(t) &= a_2M\varepsilon - (r_1 + d)y_1(t), \\
\dot{y}_2(t) &= dy_1(t) - r_2y_2(t), \\
\Delta y_1 &= p_1, \\
\Delta y_2 &= p_1,
\end{aligned}
\]
For any initial value \((y_1(0), y_2(0))\), solving (7), we can obtain
\[
\begin{aligned}
y_1(t) &= y_1^*(t) + (y_1(0) - \frac{\varepsilon a_2M}{r_1 + d})e^{\int_{0}^{t}(-r_1\alpha_1 + d)dt} + \frac{\varepsilon a_2M}{r_1 + d}t, \\
y_2(t) &= y_2^*(t) + [y_2(0) - \frac{p_2}{1 - e^{-r_2\tau}} - \frac{d p_1 (e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau})}{1 - e^{-r_1\alpha_1 + d\tau}}]e^{-r_2\tau} + \frac{y_1(0) - \frac{e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau}}{1 - e^{-r_1\alpha_1 + d\tau}}t}{r_1 + d - r_2} + \frac{e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau}}{1 - e^{-r_1\alpha_1 + d\tau}} + \frac{\varepsilon a_2M}{r_1 + d} + \frac{1}{1 - e^{-r_1\alpha_1 + d\tau}} + \frac{1}{r_1 + d - r_2} - \\
&\quad \times \left[ \frac{1}{r_2(r_1 + d - r_2)} + \frac{1}{1 - e^{-r_1\alpha_1 + d\tau}} + \frac{1}{r_2(r_1 + d - r_2)} \right] + \frac{e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau}}{1 - e^{-r_1\alpha_1 + d\tau}} + \frac{e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau}}{1 - e^{-r_1\alpha_1 + d\tau}} + \frac{e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau}}{1 - e^{-r_1\alpha_1 + d\tau}} + \frac{e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau}}{1 - e^{-r_1\alpha_1 + d\tau}} + \frac{e^{-r_2\tau} - e^{-r_1\alpha_1 + d\tau}}{1 - e^{-r_1\alpha_1 + d\tau}}.
\end{aligned}
\]
Thus, when \( t \) is large enough, we obtain
\[
\begin{align*}
    y_1(t) &< y_1^*(t) + \varepsilon_1, \\
y_2(t) &< y_2^*(t) + \varepsilon_1.
\end{align*}
\]

On the other hand, from the first equation of (1), we have \( \dot{y}_1(t) \geq -(r_1 + d)y_1(t) \). By Lemma 3, we know, for sufficiently small \( \varepsilon > 0 \), when \( t \) is large enough, the following holds
\[
\begin{align*}
    y_1(t) &> y_1^*(t) - \varepsilon_1, \\
y_2(t) &> y_2^*(t) - \varepsilon_1.
\end{align*}
\]
Thus, we can obtain \( \lim_{t \to \infty} (y_1(t), y_2(t)) = (y_1^*(t), y_2^*(t)) \). That is, the periodic solution \( (0, y_1^*(t), y_2^*(t)) \) is globally asymptotically stable. \( \square \)

4. Permanence

In this section, we investigate the permanence of system (1). We give the following result.

Theorem 2 System (1) is permanent if the following conditions are satisfied
\( (H_2) \) \quad \tau > \frac{a}{rr_2} (p_2 + \frac{dp}{r_1 + d}) > 0. 

Proof From the second equation of system (1), we have \( \dot{y}_2(t) = -(r_2 + d)y_2(t) \). Let \( m_2 = \frac{a e^{-(r_1 + d)\tau}}{1 - e^{-(r_1 + d)\tau}} \), by comparison theorem and Lemma 3, for sufficiently small \( \varepsilon_2 > 0 \), we have \( m_2 - \varepsilon_2 > 0 \). Thus, we obtain \( y_1(t) > m_2 \) when \( t \) is large enough. It is similar to find a positive constant \( m_3 = \frac{a e^{-(r_1 + d)\tau}}{1 - e^{-(r_1 + d)\tau}} \), such that \( y_2(t) > m_3 \) when \( t \) is large enough. Now we only need to find \( m_1 \), such that \( x(t) > m_1 \). We will do it in the following two steps.

(i) Let \( 0 < m_4 < \frac{r_1 - \frac{a}{r_2} (a_2 + \frac{da}{r_1 + d})}{(a + a_1 a_2 dMC) \tau} \), where
\[
C = \frac{1}{r_2 (r_1 + d)} \left( 1 + \frac{1}{1 - e^{-r_2 \tau}} \right) + \frac{1}{r_1 + d} \left( 1 + \frac{1}{1 - e^{-r_2 \tau}} \right).
\]
Take \( \varepsilon_3 \) to be small enough, such that
\[
\eta \overset{\text{def}}{=} e^{\int_0^\tau (r - am_4 - a_1 y_2^*(t) + a_2 dMC + \varepsilon_3))dt} > 1.
\]
We will prove \( x(t) < m_4 \) cannot hold for all \( t \geq 0 \). Otherwise, \( \dot{y}_1(t) < a_2 m_4 M - (r_1 + d)y_1(t) \).

Consider the comparison system
\[
\begin{align*}
\dot{y}_1(t) &= a_2 M m_4 - (r_1 + d)y_1(t), \\
\dot{y}_2(t) &= dy_1(t) - r_2 y_2(t), \\
\Delta y_1 &= p_1, \\
\Delta y_2 &= p_2,
\end{align*}
\]
\( t \neq n\tau; \)
\( t = n\tau. \)

Similarly to (7), when \( t \) is large enough, let \( t \in (n\tau, (n + 1)\tau] \). We have
\[
y_2(t) \leq y_2^*(t) + m_4 a_2 dMC + \varepsilon_3.
\]
Hence, we can obtain
\[
\dot{x}(t) \geq x(t) [r - am_4 - a_1 (y_2^*(t) + m_4 a_2 dMC + \varepsilon_3)].
\]
Integrating (9) on \((n\tau, (n+1)\tau]\), we have
\[
x((n+1)\tau) \geq x(n\tau) e^{\int_{n\tau}^{(n+1)\tau}(r-a_1(y_2(t)+m_4a_2dMC+\varepsilon_3))dt} = x(n\tau)\eta.
\]
Then \(x((n+k)\tau) \geq x(n\tau)\eta^k \to \infty\) as \(k \to \infty\), which is a contradiction with the conclusion of Lemma 2. Hence, there exists a \(t_1 > 0\), such that \(x(t_1) \geq m_4\).

(ii) If \(x(t) \geq m_4\) for all \(t \geq t_1\), then our aim is obtained. Hence, we only need to consider those solutions which leave the region \(\{(x, y_1, y_2) \in \mathbb{R}^3 : x \geq m_4\}\) and reenter it again. Let \(t^* = \inf_{t \geq t_1}\{x(t) < m_4\}\). Then \(x(t) > m_4\) for \(t \in [t_1, t^*)\) and \(x(t^*) = m_4\) since \(x(t)\) is continuous. Suppose \(t^* \in [n_1\tau, (n_1+1)\tau]\), \(n_1 \in \mathbb{Z}_+\). Select \(n_2, n_3 \in \mathbb{Z}_+,\) such that
\[
n_2\tau > -\frac{1}{r^2} \ln \frac{\varepsilon_3}{\sigma_1} e^{n_2\tau\sigma_1\eta^{n_3}} > 1
\]
where \(\sigma_1 = r - a_1 M < 0\),
\[
C_1 = |y_2(0) - \frac{p_2}{1 - e^{-r\tau}} - \frac{dp_1}{1 - e^{-r\tau}}\left(1 - e^{-r\tau}\right)\left(1 - e^{-(r_1+d)\tau}\right) + \frac{d(y_1(0) - \frac{p_1}{1 - e^{-(r_1+d)\tau}}\left(1 - e^{-r\tau}\right)\left(1 - e^{-(r_1+d)\tau}\right)\left(1 - e^{-r\tau}\right)}{1 - e^{-r\tau}}\left(1 - e^{-r\tau}\right)}\right).
\]
Let \(T = (n_2 + n_3)\tau\). We claim there must be a \(t_2 \in [t^*, t^* + T]\) such that \(x(t_2) \geq m_4\). Otherwise, \(t \in [t^*, t^* + T]\), \(y_1(t) \leq a_2 m_4 M - (r_1 + d)y_1(t)\). Similarly to (7), when \(t \in [t^* + n_2\tau, t^* + T]\), we have
\[
|y_2(t) - y_2^*(t) - m_4a_2dMC| \leq C_1 e^{-n_2r_2\tau}.
\]
Hence \(y_2(t) \leq y_2^*(t) + m_4a_2dMC + \varepsilon_3\). So as in step (1), we have \(x(t^* + T) \geq x(t^* + n_2\tau)\eta^{n_3}\).

On the other hand, by (1), we can obtain
\[
\dot{x}(t) \geq x(t)(r - a_1 M - a_1 M) = \sigma_1 x(t).
\]
Integrating (10) on \([t^*, t^* + n_2\tau]\), we have \(x(t^* + n_2\tau) \geq m_4e^{\sigma_1 n_2\tau}\). This implies \(x(t^* + T) \geq m_4e^{\sigma_1 n_2\tau}\eta^{n_3} > m_4\), which is a contradiction. Let \(\bar{t} = \inf_{t \geq t^*}\{x(t) \geq m_4\}\). Then \(x(\bar{t}) \geq m_4\). For \(t \in [t^*, \bar{t}]\), since \(\dot{x}(t) \geq \sigma_1 x(t)\), we have \(x(t) \geq x(t^*)e^{\sigma_1(t-t^*)} \geq m_4e^{\sigma_1(l+n_2+n_3)\tau} \geq m_4\). For \(t > \bar{t}\), the same arguments can be continued since \(x(\bar{t}) \geq m_4\). Hence \(x(t) \geq m_1\) for \(t \geq t_1\). Thus system (1) is permanent when the condition (H2) is satisfied. \(\square\)

5. Numerical simulations

In this section, we investigate the impulsive effect on system (1). Now, let us give some numerical simulations. For system (1), we choose \(a = 1\), \(a_1 = 2\), \(a_2 = 4\), \(d = 8\), \(m = 1\), \(r = 2\), \(r_1 = 0.5\) and \(r_2 = 0.76\). Numerical simulations are shown in Figures 1–4.

By Theorem 1, as \(p_1 = 0.1\) and \(p_2 = 0.2\), we know that the pest-eradication periodic solution \((0, y_1^*(t), y_2^*(t))\) is globally asymptotically stable when \(\tau < 0.129\). From Fig.1, clearly, as \(\tau = 0.1\), a typical pest-eradication periodic solution of system (1) is shown. As \(\tau = 25\), \(p_1 = 0.1\) and \(p_2\) increases from 0.001 to 12, there are typical bifurcation diagrams for the pest population and immature predator population of system (1) in Figure 2. The resulting bifurcation diagrams
clearly show that system (1) has rich dynamics including period-doubling bifurcation, periodic halving bifurcation, symmetry-breaking pitchfork bifurcation and chaos. Figures 3 and 4 give the time-series of the system (1) with $p_1 = 0.1$, $p_2 = 3.8$ and $p_1 = 0.1$, $p_1 = 1$, respectively. A typical periodic solution of system (1) is shown in Figure 3 and a chaotic oscillation is shown in Figure 4.

The similar results can be obtained as $p_2 = 0.1$ and $p_1$ increases from 0.001 to 12.

![Figure 1](image1.png)  
**Figure 1** Time-series of system (1) with $p_1 = 0.1$, $p_2 = 0.2$ and $\tau = 0.1$

![Figure 2](image2.png)  
**Figure 2** Bifurcation diagrams with $\tau = 25$, $p_1 = 0.1$ and $p_2$ over $[0.01, 12]$

![Figure 3](image3.png)  
**Figure 3** Time-series of system (1) with $p_1 = 0.1$ and $p_2 = 3.8$

6. Discussion

In this paper, we discuss a stage-structured predator-prey system with impulsive control strategy in detail. We have proved that there exists a globally stable pest-eradication periodic
solution. When the stability of pest-eradication periodic solution is lost, we have shown that the system is permanent. Numerical results show there is a characteristic sequence of bifurcations, leading to a chaotic dynamics, which implies that dynamical behaviors of the system concerning impulsive control strategy are very complex.

![Figure 4 Time-series of system (1) with $p_1 = 0.1$ and $p_2 = 1$](image)

From Theorem 1, we know that the pest-eradication periodic solution $(0, y_1^*(t), y_2^*(t))$ is globally stable if $\tau < \frac{\alpha_1}{\tau_2(a + mr)}(p_2 + \frac{a_1}{r_1 + d}) = \tau_{\text{max}}$. Therefore, in order to drive the pest to extinction, we can take impulsive control strategy according to cost of releasing natural enemies such that $\tau < \tau_{\text{max}}$.

References


