On the Reduced Minimum Modulus of Projections and the Angle between Two Subspaces

Xiu Hong SUN¹, Yuan LI²,*

1. School of Science, Xi’an University of Science and Technology, Shaanxi, 710054, P. R. China; 2. College of Mathematics and Information Science, Shaanxi Normal University, Shaanxi 710062, P. R. China

Abstract Let $M$ and $N$ be nonzero subspaces of a Hilbert space $H$, and $P_M$ and $P_N$ denote the orthogonal projections on $M$ and $N$, respectively. In this note, an exact representation of the angle and the minimum gap of $M$ and $N$ is obtained. In addition, we study relations between the angle, the minimum gap of two subspaces $M$ and $N$, and the reduced minimum modulus of $(I - P_N)P_M$.

Keywords subspace; angle between two subspaces; reduced minimum modulus.

Document code A

MR(2010) Subject Classification 46C05; 46C07

Chinese Library Classification O177.2

1. Introduction

Throughout this note, a subspace is a closed linear manifold of a separable Hilbert space $H$ with inner product and norm denoted by $\langle x, y \rangle$ and $\|x\| = \sqrt{\langle x, x \rangle}$, respectively. If $M$ is a subspace of $H$, the orthogonal complement of $M$ is denoted by $M^\perp$ and the orthogonal projection on $M$ is denoted by $P_M$. In recent years, the variety of quantities involving two subspaces have been studied by a number of researchers in the wide literatures [2–11]. In this note, using the technique of block-operators, some results about minimum gap and the angle between two closed subspaces of a Hilbert space are improved which are obtained by Deng in [3] and other results concerning two subspaces of a Hilbert space are obtained. The angle [6, 7, 9] between $M$ and $N$ is an angle in $[0, \frac{\pi}{2}]$ whose cosine is defined by

$$c(M, N) = \sup\{|\langle x \rangle y | : x \in M \cap (M \cap N)^\perp, y \in N \cap (M \cap N)^\perp, \|x\| = \|y\| = 1\}.$$ (1)

By this formula $c(M, N)$ is defined only when $M$ is not a subspace of $N$ and $N$ is not a subspace of $M$. If $M \subseteq N$ or $N \subseteq M$, we let $c(M, N) = 0$. The minimal angle [7] between $M$ and $N$ is an angle in $[0, \frac{\pi}{2}]$ whose cosine is defined by

$$c_0(M, N) = \sup\{|\langle x \rangle y | : x \in M, y \in N, \|x\| = \|y\| = 1\}.$$ (2)

Received December 20, 2008; Accepted June 30, 2009

Supported by the National Natural Science Foundation of China (Grant No. 10871224) and the Fundamental Research Funds for the Central Universities (Grant No. GK 200902049).

* Corresponding author

E-mail address: liyuan0401@yahoo.com.cn (Y. LI)
Recall that the minimum gap $\gamma(\mathcal{M}, \mathcal{N})$ between two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space has been defined [3, 9] by

$$\gamma(\mathcal{M}, \mathcal{N}) = \inf_{x \in \mathcal{M}, x \notin \mathcal{N}} \frac{\text{dist}(x, \mathcal{N})}{\text{dist}(x, \mathcal{M} \cap \mathcal{N})}. \quad (3)$$

By this formula $\gamma(\mathcal{M}, \mathcal{N})$ is defined only when $\mathcal{M}$ is not a subspace of $\mathcal{N}$. If $\mathcal{M} \subseteq \mathcal{N}$, we set $\gamma(\mathcal{M}, \mathcal{N}) = 1$. Obviously, $\gamma(\mathcal{M}, \mathcal{N}) = 1$, if $\mathcal{N} \subseteq \mathcal{M}$.

Before proving the main results in this paper, let us introduce some notations and terminology which are used in the later. The set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For an operator $A \in \mathcal{B}(\mathcal{H})$, the adjoint, the range, the null-space and the spectrum of $A$ are denoted by $A^*$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\sigma(A)$, respectively. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $A = A^*$. An operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $(Ax, x) \geq 0$ for $x \in \mathcal{H}$. If $A$ is a positive operator, the unique square root of $A$ is denoted by $A^\frac{1}{2}$. An operator $A$ is said to be a contraction (strict contraction) if $\|A\| \leq 1$ ($\|A\| < 1$). The reduced minimum modulus $\gamma(A)$ of $A \in \mathcal{B}(\mathcal{H})$ (see [1, 9]) is defined by

$$\gamma(A) = \begin{cases} \inf \{\|Ax\| : \text{dist}(x, \mathcal{N}(A)) = 1\}, & A \neq 0; \\ 0, & A = 0. \end{cases}$$

It is well known that for $A \neq 0$, $\mathcal{R}(A)$ is closed if and only if $\gamma(A) > 0$.

2. Main results

**Lemma 1** ([4, 8]) Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_\mathcal{M}$ and $P_\mathcal{N}$ denote the orthogonal projections on $\mathcal{M}$ and $\mathcal{N}$, respectively, then $P_\mathcal{M}$ and $P_\mathcal{N}$ have the operator matrices

$$P_\mathcal{M} = I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus I_5 \oplus 0I_6 \quad (4)$$

and

$$P_\mathcal{N} = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q & Q^\frac{1}{2}(I_5 - Q)^\frac{1}{2} \\ D^*Q^\frac{1}{2}(I_5 - Q)^\frac{1}{2} & D^*(I_5 - Q)D \end{pmatrix} \quad (5)$$

where $\mathcal{H} = \bigoplus_{i=1}^{6} \mathcal{H}_i$, respectively, where $\mathcal{H}_1 = \mathcal{M} \cap \mathcal{N}$, $\mathcal{H}_2 = \mathcal{M} \cap \mathcal{N}^\perp$, $\mathcal{H}_3 = \mathcal{M}^\perp \cap \mathcal{N}$, $\mathcal{H}_4 = \mathcal{M}^\perp \cap \mathcal{N}^\perp$, $\mathcal{H}_5 = \mathcal{M} \ominus (\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\mathcal{H}_6 = \mathcal{H} \ominus (\bigoplus_{j=1}^{5} \mathcal{H}_j)$, $Q$ is a positive contraction on $\mathcal{H}_5$, $0$ and $1$ are not eigenvalues of $Q$, and $D$ is a unitary from $\mathcal{H}_6$ onto $\mathcal{H}_5$. $I_i$ is the identity on $\mathcal{H}_i$, $i = 1, \ldots, 5$.

For convenience, in the sequel, we always assume that $P_\mathcal{M}$ and $P_\mathcal{N}$ have the operator matrices (4) and (5), also the zero operator on $\mathcal{H}_i$ is denoted by $0I_i$, $i = 1, \ldots, 6$.

First, we give some necessary and sufficient conditions for $\mathcal{H}_5 = \mathcal{H}_6 = \{0\}$.

**Lemma 2** Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. The following statements are equivalent:

(a) $P_\mathcal{M}$ and $P_\mathcal{N}$ commute: $P_\mathcal{M}P_\mathcal{N} = P_\mathcal{N}P_\mathcal{M}$;

(b) $P_\mathcal{M}P_\mathcal{N} = P_{\mathcal{N} \cap \mathcal{M}}$;

(c) $P_\mathcal{M}P_\mathcal{N}$ is an orthogonal projections;

(d) $P_\mathcal{M}P_\mathcal{N}$ is an idempotent;
Lemma 1 that
\[ (e) \quad P_M P_N P_M = P_N P_M P_N; \]
\[ (f) \quad H = \mathcal{M} \cap \mathcal{N} \oplus \mathcal{M} \cap \mathcal{N}^\perp \oplus \mathcal{M}^\perp \cap \mathcal{N} \oplus \mathcal{N}^\perp \cap \mathcal{M}^\perp; \]
\[ (g) \quad \mathcal{M} = \mathcal{M} \cap \mathcal{N} \oplus \mathcal{M} \cap \mathcal{N}^\perp. \]

**Proof** Since \( \|P_M P_N\| \leq 1 \), it is clear that (c)\( \Leftrightarrow \) (d).

(c)\( \Rightarrow \) (f). By Lemma 1, \( P_M \) and \( P_N \) have the operator matrices (4) and (5). It is easy to calculate that
\[ P_M P_N = I_1 \oplus 0 I_2 \oplus 0 I_3 \oplus 0 I_4 \begin{pmatrix} Q & Q^2/2 (I_5 - Q)^{3/2} D \\ 0 & 0 \end{pmatrix}, \]
and
\[ (P_M P_N)^2 = I_1 \oplus 0 I_2 \oplus 0 I_3 \oplus 0 I_4 \begin{pmatrix} Q^2 & Q^2/2 (I_5 - Q)^{3/2} D \\ 0 & 0 \end{pmatrix}. \]

Therefore, if \( H_5 \neq \{0\} \), then \( Q^2 = Q \), so \( \sigma(Q) = \{0, 1\} \), hence 0 and 1 are eigenvalues of \( Q \). It is a contradiction to Lemma 1, so \( H_5 = \{0\} \), then \( H_6 = \{0\} \). Thus \( H = \mathcal{M} \cap \mathcal{N} \oplus \mathcal{M} \cap \mathcal{N}^\perp \oplus \mathcal{M}^\perp \cap \mathcal{N} \oplus \mathcal{N}^\perp \cap \mathcal{M}^\perp \).

(e)\( \Rightarrow \) (f). By a similar calculation as above, we have \( P_M P_N P_M = P_N P_M P_N \) which implies \( H_5 = H_6 = \{0\} \).

It is obvious that (f)\( \Rightarrow \) (g)\( \Rightarrow \) (a)\( \Leftrightarrow \) (b)\( \Leftrightarrow \) (c). \( \square \)

The following lemma was obtained in [1].

**Lemma 3** ([1]) Let \( T \in B(H) \). Then
\[ \gamma(T) = \gamma(T^*) = (\inf \{\sigma(T T^*) \setminus \{0\}\})^{\frac{1}{2}} = (\inf \{\sigma(T^* T) \setminus \{0\}\})^{\frac{1}{2}}. \]

From above lemmas, we give the specific representation of \( \gamma(P_M P_N) \).

**Theorem 4** Let \( \mathcal{M} \) and \( \mathcal{N} \) be nonzero subspaces of \( H \). Then
\[ \gamma(P_M P_N) = \begin{cases} 1, \quad \text{if } P_M P_N \text{ is a nonzero orthogonal projection;} \\
0, \quad \text{if } P_M P_N = 0; \\
(1 - \|I - Q\|)^{\frac{1}{2}}, \quad \text{if } P_M P_N \text{ is not an orthogonal projection}. \end{cases} \]

**Proof** If \( P_M P_N \) is not an orthogonal projection, then \( H_5 \neq \{0\} \) and \( H_6 \neq \{0\} \). It follows from Lemma 1 that
\[ P_M P_N (P_M P_N)^* = I_1 \oplus 0 I_2 \oplus 0 I_3 \oplus 0 I_4 \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}. \]

Since 0 is not an eigenvalue of \( Q \), 0 is not an isolated point of \( \sigma(Q) \). Hence by Lemma 2,
\[ \gamma(P_M P_N) = (\inf \{\sigma(Q) \setminus \{0\}\})^{\frac{1}{2}} = (\inf \{\sigma(Q)\})^{\frac{1}{2}}. \]

Thus
\[ \gamma(P_M P_N) = \inf \{\lambda \in \mathbb{C} : \lambda \in \sigma(Q)\}^{\frac{1}{2}} = (1 - \sup \{\lambda \in \mathbb{C} : \lambda \in \sigma(I_5 - Q)\})^{\frac{1}{2}} = (1 - \|I_5 - Q\|)^{\frac{1}{2}}. \quad \square \]

It is well-known that
\[ \sup \{\sigma(T)\} = \lim_{n \to \infty} \|(T^n)\|^{\frac{1}{n}}, \]
and if $T$ is invertible, then
\[ \lim_{k \to \infty} \gamma(T^k)^\frac{1}{k} = \inf\{\sigma(T)\}. \]
Similarly, we have following conclusion for $T = P_M P_N$.

**Theorem 5** Let $T \neq 0$ be product of two orthogonal projections. Then
\[ \lim_{k \to \infty} \gamma(T^k)^\frac{1}{k} = \inf\{\sigma(T) \setminus \{0\}\}. \]

**Proof** If $T$ is an orthogonal projection, then $\lim_{k \to \infty} \gamma(T^k)^\frac{1}{k} = 1$, the conclusion is clear.

If $T$ is not an orthogonal projection, then let $T = P_M P_N$, where $P_M$ and $P_N$ have the operator matrices (4) and (5). By Lemma 1, $H$ is invertible, then $\gamma(T^k)^\frac{1}{k} = \inf\{\sigma(T) \setminus \{0\}\}$. It is easy to calculate that
\[
(P_M P_N)^k = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q^k & Q^{2k-1} I_2 - Q \frac{1}{2} I_1 \end{pmatrix} \]
and
\[
(P_M P_N)^k((P_M P_N)^*)^k = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q^{2k-1} & 0 \\ 0 & 0 \end{pmatrix}. \]
Thus
\[
\gamma((P_M P_N)^k) = \inf\{\sigma(Q^{2k-1}) \setminus \{0\}\}\]
\[= (\inf\{\sigma(Q)^{2k-1} \setminus \{0\}\})^{\frac{1}{k}} \text{ (by Spectra Mapping Theorem)} \]
\[= (\inf\{\sigma(Q)^{k-1}\})^{\frac{1}{k}} \text{ (since 0 is not an eigenvalues of } Q^{2k-1} \text{)} \]
\[= (\inf\{\sigma(Q)\})^{\frac{2k-1}{k}}. \]
Hence $\lim_{k \to \infty} \gamma(T^k)^\frac{1}{k} = \inf\{\sigma(Q)\}$. It is easy to see that
\[\sigma(Q) \subseteq \sigma(T) = \sigma(P_M P_N) \subseteq \{1, 0\} \cup \sigma(Q),\]
so $\inf\{\sigma(T) \setminus \{0\}\} = \inf\{\sigma(Q) \setminus \{0\}\} = \inf\{\sigma(Q)\}$. Therefore,
\[\lim_{k \to \infty} \gamma(T^k)^\frac{1}{k} = \inf\{\sigma(T) \setminus \{0\}\}. \]

In Lemma 2.10 of [7], the following results were obtained. To make this work complete, we include a proof.

**Lemma 6** Let $M$ and $N$ be nonzero subspaces of $H$. Then
\[ c_0(M, N) = \|P_M P_N\| = \|P_M P_N P_M\|^{\frac{1}{k}}, \]
and
\[ c(M, N) = \|P_M P_N (P_M \cap N)^\perp\|. \]

**Proof**
\[ c_0(M, N) = \sup\{\|x\| : x \in M, y \in N \text{ and } \|x\| = \|y\| = 1\} \]
\[ = \sup\{\|P_M x, P_N y\| : x, y \in H \text{ and } \|x\| = \|y\| = 1\} \]
\[ = \sup\{\|x, P_M P_N y\| : x, y \in H \text{ and } \|x\| = \|y\| = 1\} \]
Theorem 8
Let \( M \) and \( N \) be nonzero subspaces of \( \mathcal{H} \). Then
\[
c(\mathcal{M}, \mathcal{N}) = c_0(\mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^\perp, \mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^\perp)
\]
\[
= \| P_M P_N \|.
\]
From Lemmas 6 and 2, it is easy to see that if \( P_M P_N \) is an orthogonal projection, then \( c(\mathcal{M}, \mathcal{N}) = 0 \). If \( P_M P_N \) is not an orthogonal projection, then \( \mathcal{H}_5 \neq 0 \) and \( \mathcal{H}_6 \neq 0 \). Note that
\[
P_{(\mathcal{M} \cap \mathcal{N})^\perp} = 0I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus I_6,
\]
so
\[
c(\mathcal{M}, \mathcal{N}) = \| P_M P_N P_{(\mathcal{M} \cap \mathcal{N})^\perp} \| = \| P_M P_N P_{(\mathcal{M} \cap \mathcal{N})^\perp} P_{(\mathcal{M} \cap \mathcal{N})^\perp} \| ^{\frac{1}{2}} = \| Q \| ^{\frac{1}{2}}.
\]
From the relation of \( c(\mathcal{M}, \mathcal{N}) \) and \( c_0(\mathcal{M}, \mathcal{N}) \), the expression of \( c_0(\mathcal{M}, \mathcal{N}) \) is clear. □

The following theorem is one of our main results.

**Theorem 8** Let \( M \) and \( N \) be nonzero subspaces of \( \mathcal{H} \). Then
1. If \( M \subseteq N \), then \( \gamma(P_N - P_M) = c(\mathcal{M}, \mathcal{N}) = 0 \).
2. If \( M \nsubseteq N \), then \( \gamma^2(P_N - P_M) + c^2(\mathcal{M}, \mathcal{N}) = 1 \).

**Proof** (a) is clear.

(b) Case 1. If \( P_M P_N \) is an orthogonal projection, then \( P_{N - P_M} \) is a nonzero orthogonal projection, so \( \gamma(P_{N - P_M}) = 1 \) and \( c(\mathcal{M}, \mathcal{N}) = 0 \), by Lemma 7.

Case 2. If \( P_M P_N \) is not an orthogonal projection, it follows from Lemma 2 that \( \mathcal{H}_5 \neq 0 \), then \( \mathcal{H}_6 \neq 0 \). By Lemma 1, it is easy to calculate that
\[
P_{N - P_M} = 0I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus \left( D^*(I_5 - Q)^{\frac{1}{2}} Q^{\frac{1}{2}} - Q^{\frac{1}{2}} \right)
\]
and
\[
P_{N - P_M}(P_{N - P_M})^* = 0I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus \left( 1 - Q^{\frac{1}{2}} \right).
\]
Thus by Lemma 3,
\[
\gamma(P_{N - P_M}) = \inf \{ (1 - Q) \setminus \{0\} \} = (1 - \sup \{ \lambda \in \mathbb{C} : \lambda \in \sigma(Q) \})^{\frac{1}{2}}
\]
By Lemma 3, $\gamma^2(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) + c^2(\mathcal{M}, \mathcal{N}) \leq 1$. □

Consequently, we obtain some results of [2] and [7].

**Corollary 9** ([7, Theorems 2.15, 2.16]) Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then

(a) $c(\mathcal{M}, \mathcal{N}) = c(\mathcal{M}^\perp, \mathcal{N}^\perp)$;

(b) If $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{H}$, then $c_0(\mathcal{M}, \mathcal{N}) = c_0(\mathcal{M}^\perp, \mathcal{N}^\perp)$.

**Proof** (a) Case 1. If $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{N}^\perp \subseteq \mathcal{M}^\perp$, so by Lemma 7, $c(\mathcal{M}, \mathcal{N}) = c(\mathcal{M}^\perp, \mathcal{N}^\perp) = 0$.

Case 2. If $\mathcal{M} \not\subseteq \mathcal{N}$, then $\mathcal{N}^\perp \not\subseteq \mathcal{M}^\perp$, by Theorem 8,

$$\gamma^2(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) + c^2(\mathcal{M}, \mathcal{N}) = 1$$

By Lemma 3, $\gamma(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) = \gamma(P_{\mathcal{M}}P_{\mathcal{N}^\perp})$, so $c(\mathcal{M}, \mathcal{N}) = c(\mathcal{N}^\perp, \mathcal{M}^\perp) = c(\mathcal{M}^\perp, \mathcal{N}^\perp)$.

(b) Since $\mathcal{M} \cap \mathcal{N} = \{0\}$, it is obvious that $c_0(\mathcal{M}, \mathcal{N}) = c(\mathcal{M}, \mathcal{N})$. It follows from $\mathcal{M} + \mathcal{N} = \mathcal{H}$ that $\mathcal{M}^\perp \cap \mathcal{N}^\perp = \{0\}$, so $c_0(\mathcal{M}^\perp, \mathcal{N}^\perp) = c(\mathcal{M}^\perp, \mathcal{N}^\perp)$. According to (a), $c_0(\mathcal{M}, \mathcal{N}) = c_0(\mathcal{M}^\perp, \mathcal{N}^\perp)$. □

**Corollary 10** ([7, Theorem 2.13]) Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then the following statements are equivalent:

(a) $c(\mathcal{M}, \mathcal{N}) < 1$;

(b) $\mathcal{M} + \mathcal{N}$ is closed;

(c) $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed.

**Proof** If $\mathcal{M} \subseteq \mathcal{N}$, then the conclusion is clear. In the following proof, we assume $\mathcal{M} \not\subseteq \mathcal{N}$. It is easy to see that $\mathcal{M} + \mathcal{N} = \mathcal{N} + P_{\mathcal{N}^\perp}(\mathcal{M})$, where $P_{\mathcal{N}^\perp}(\mathcal{M}) := \{P_{\mathcal{N}^\perp}y : y \in \mathcal{M}\}$. Hence $\mathcal{M} + \mathcal{N}$ is closed if and only if $\mathcal{R}(P_{\mathcal{N}^\perp}(\mathcal{M}))$ is closed. It is well-known that $\mathcal{R}(P_{\mathcal{N}^\perp}(\mathcal{M}))$ is closed if and only if $\gamma(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) > 0$. Therefore, it follows from Theorem 8 that (a)$\iff$ (b). From Corollary 9, $c(\mathcal{M}, \mathcal{N}) < 1 \iff c(\mathcal{M}^\perp, \mathcal{N}^\perp) < 1 \iff \mathcal{M}^\perp + \mathcal{N}^\perp$ is closed. □

The following result is fundamental in [7]. A technical proof has been given in [7]. Here, we give a simple proof.

**Corollary 11** ([7, Lemma 2.14]) Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. If $c_0(\mathcal{M}, \mathcal{N}) < 1$, then for any closed subspace $X$ of $\mathcal{H}$ which contains $\mathcal{M} + \mathcal{N}$, we have

$$c_0(\mathcal{M}, \mathcal{N}) \leq c_0(\mathcal{M}^\perp \cap X, \mathcal{N}^\perp \cap X).$$

**Proof** For convenience, we divide proof into three steps.

Step 1. If $\mathcal{M}^\perp \cap \mathcal{N}^\perp \cap X \neq \{0\}$, then $c_0(\mathcal{M}^\perp \cap X, \mathcal{N}^\perp \cap X) = 1$, so $c_0(\mathcal{M}, \mathcal{N}) \leq c_0(\mathcal{M}^\perp \cap X, \mathcal{N}^\perp \cap X)$.

Step 2. Let $X = \mathcal{H}$. Since $c_0(\mathcal{M}, \mathcal{N}) < 1$, we have $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N}$ is closed. If $\mathcal{M}^\perp \cap \mathcal{N}^\perp \neq \{0\}$, then $c_0(\mathcal{M}^\perp \cap X, \mathcal{N}^\perp \cap X) = c_0(\mathcal{M}^\perp, \mathcal{N}^\perp) = 1$, so $c_0(\mathcal{M}, \mathcal{N}) \leq c_0(\mathcal{M}^\perp \cap X, \mathcal{N}^\perp \cap X)$. 
If $M^\perp \cap N^\perp = \{0\}$, then $M + N = H$, since $M + N$ is closed. It follows from Corollary 9 that $c_0(M, N) = c_0(M^\perp \cap X, N^\perp \cap X)$.

Step 3. If $M^\perp \cap N^\perp \cap X = \{0\}$, then $(M + N)^\perp \cap X = \{0\}$. Therefore, $X = M + N$, since $X \supseteq M + N$ and $M + N$ is closed. It is easy to see that $c_0(M, N) = c_0(M \cap X, N \cap X)$, then we may replace $H$ by $X$. It follows from Step 2 that $c_0(M, N) \leq c_0(M^\perp \cap X, N^\perp \cap X)$. □

The following result has been proved in [2, 7]. As an application of Theorem 8, we give an alternative proof.

**Corollary 12 ([2, 7])** If $A$ and $B$ are bounded operators on $H$ with closed ranges, then the following statements are equivalent:

(a) $AB$ has closed range;
(b) $c(\mathcal{R}(B), \mathcal{N}(A)) < 1$;
(c) $\mathcal{R}(B) + \mathcal{N}(A)$ is closed.

**Proof** If $AB = 0$, then the conclusion is clear. In the following proof, assume that $AB \neq 0$. Since $\mathcal{R}(A)$ is closed, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{N}(A)^\perp \oplus \mathcal{N}(A) \to \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp,$$

where $A_1$ is invertible from $\mathcal{N}(A)^\perp$ onto $\mathcal{R}(A)$. It is easy to see that

$$\mathcal{R}(AB) = \mathcal{R}(A_1P_{\mathcal{N}(A)^\perp}P_{\mathcal{R}(B)}).$$

Since $A_1$ is invertible, $\mathcal{R}(AB)$ is closed $\iff \mathcal{R}(P_{\mathcal{N}(A)^\perp}P_{\mathcal{R}(B)})$ is closed $\iff \gamma(P_{\mathcal{N}(A)^\perp}P_{\mathcal{R}(B)}) > 0 \iff c(\mathcal{R}(B), \mathcal{N}(A)) < 1$, by Theorem 8. □

The following theorem is our another main result which is an extension of Theorem 8 of [3].

**Theorem 13** Let $M$ and $N$ be nonzero subspaces of $H$. Then

$$\gamma(M, N) = \begin{cases} 1, & \text{if } P_MP_N \text{ is an orthogonal projection;} \\ (1 - \|Q\|)^{\frac{1}{2}}, & \text{if } P_MP_N \text{ is not an orthogonal projection.} \end{cases}$$

Especially, $M + N$ is closed if and only if $\gamma(M, N) > 0$.

**Proof** If $P_MP_N$ is an orthogonal projection, then $H_5 = H_6 = 0$, so

$$P_M = I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4, \quad \text{and} \quad P_N = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4.$$

**Case 1** If $M \cap N^\perp = \{0\}$, then $M \subseteq N$, by the definition of $\gamma(M, N)$, we have $\gamma(M, N) = 1$.

**Case 2** If $M^\perp \cap N = \{0\}$, then $M \supseteq N$, so $\gamma(M, N) = 1$.

**Case 3** If $M^\perp \cap N \neq \{0\}$ and $M \cap N^\perp \neq \{0\}$, let $x \in M \setminus N$. Then $x = x_1 + x_2$, where $x_1 \in M \cap N$ and $0 \neq x_2 \in M \cap N^\perp$, since $P_MP_N$ is an orthogonal projection. It is easy to see that

$$\text{dist}(x, N) = \inf\{\|x - y\| : y \in N\} = \inf\{\|x_2 - y\| : y \in N\} = \|x_2\|,$$

$$\text{dist}(x, M \cap N) = \inf\{\|x - y\| : y \in M \cap N\} = \inf\{\|x_2 - y\| : y \in M \cap N\} = \|x_2\|.$$
Hence $\gamma(\mathcal{M}, \mathcal{N}) = 1$.

If $P_{\mathcal{M}} P_{\mathcal{N}}$ is not an orthogonal projection, then $\mathcal{H}_5 \neq 0$ and $\mathcal{H}_6 \neq 0$. For a vector $x \in \mathcal{M} \setminus \mathcal{N}$, $x$ has the decomposition $x = x_1 + x_2 + x_5$ with $x_i \in \mathcal{H}_i$, $i = 1, 2, 5$, then $\|x_2\|^2 + \|x_5\|^2 \neq 0$, so

$$\gamma(\mathcal{M}, \mathcal{N}) = \inf_{x \in \mathcal{M}, x \notin \mathcal{N}} \frac{\text{dist}(x, \mathcal{N})}{\text{dist}(x, \mathcal{M} \cap \mathcal{N})} = \inf_{x \in \mathcal{M}, x \notin \mathcal{N}} \frac{\sqrt{\|x_2\|^2 + \|(I_5 - Q)\frac{x}{5}\|_5^2}}{\|x_2\|^2 + \|x_5\|^2}$$

$$= \inf_{x \in \mathcal{M}, x \notin \mathcal{N}} \frac{\|(I_5 - Q)\frac{x}{5}\|_5}{\|x_5\|} = \inf_{x_5 \in \mathcal{H}_5 \setminus \{0\}} \frac{\|(I_5 - Q)\frac{x}{5}\|_5}{\|x_5\|}$$

(note that $x_5 \in \mathcal{H}_5$ implies $x_5 \in \mathcal{M} \setminus \mathcal{N}$)

$$= \gamma((I_5 - Q)\frac{1}{2}),$$

since $\mathcal{N}(I_5 - Q) = \{0\}$. It follows from Lemma 3 that

$$\gamma((I_5 - Q)\frac{1}{2}) = \inf \{\sigma(I_5 - Q) \setminus \{0\}\}^{\frac{1}{2}} = \inf \{\sigma(I_5 - Q)\}^{\frac{1}{2}}$$

$$= (1 - \sup \{\lambda \in \mathbb{C} : \lambda \in \sigma(Q)\})^{\frac{1}{2}} = (1 - \|Q\|)^{\frac{1}{2}}.$$

By Corollary 10 and Corollary 7, $\mathcal{M} + \mathcal{N}$ is closed $\iff c(\mathcal{M}, \mathcal{N}) < 1 \iff \|Q\| < 1 \iff \gamma(\mathcal{M}, \mathcal{N}) > 0$. 

Combining Theorem 8 and Theorem 13, we obtain the following result.

**Corollary 14** Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then

(a) If $\mathcal{M} \subseteq \mathcal{N}$, then $\gamma(P_{\mathcal{N}}, P_{\mathcal{M}}) = 0$ and $\gamma(\mathcal{M}, \mathcal{N}) = 1$;

(b) If $\mathcal{M} \nsubseteq \mathcal{N}$, then $\gamma(P_{\mathcal{N}}, P_{\mathcal{M}}) = \gamma(\mathcal{M}, \mathcal{N})$;

(c) $\gamma(\mathcal{M}, \mathcal{N}) = \gamma(\mathcal{N}^\perp, \mathcal{M}^\perp)$.

**References**


