Note about Fixed Points of Scott Continuous Self-Mappings

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Abstract  It is discussed in this paper that under what conditions, for a continuous domain $L$, there is a Scott continuous self-mapping $f : L \to L$ such that the set of fixed points $\text{fix}(f)$ is not continuous in the ordering induced by $L$. For any algebraic domain $L$ with a countable base and a smallest element, the problem presented by Huth is partially solved. Also, an example is given and shows that there is a bounded complete domain $L$ such that for any Scott continuous stable self-mapping $f$, $\text{fix}(f)$ is not the retract of $L$.

Keywords  Scott continuous; fixed points; stable mapping.

Document code  A  
MR(2010) Subject Classification  54A10; 06B35  
Chinese Library Classification  O189.1; O153.1

1. Introduction and preliminaries

A poset $L$ is called DCPO if every directed subset of $L$ has the supremum. For all $x, y \in L$, $x \ll y$ if and only if for every directed subset $E$ of $L$ satisfying $y \leq \bigvee E$, there is $e \in E$ such that $x \leq e$. For any $x \in L$, we write $\downarrow x = \{y \in L | y \ll x\}$. A poset is called to be continuous if for any $x \in L$, $\downarrow x$ is directed and $x = \bigvee \downarrow x$. A continuous DCPO is also called a continuous domain. For any $x \in L$, $x$ is called a compact element if $x \ll x$. The set of all compact elements in $L$ is denoted by $K(L)$. A DCPO $L$ is called an algebraic domain if $\downarrow x \cap K(L)$ is directed and $\text{sup}(\downarrow x \cap K(L)) = x$ for every $x \in L$. A DCPO $L$ is called $L$-domain if for any $x \in L$, $\downarrow x$ is a complete lattice. A DCPO $L$ is called a bounded complete domain if each pair of elements of $L$ has a least upper bound. Suppose that $D, L$ are DCPO. If the mapping $f : D \to L$ preserves the supremum of directed subsets, then $f$ is called Scott continuous. If there are Scott continuous mappings $r : D \to L$ and $s : L \to D$ such that $r \circ s = \text{id}_L$, then $L$ is called the retract of $D$. If the Scott continuous self-mapping $f : L \to L$ satisfies $f \circ f = f$, then $\text{fix}(f) = \{x \in L | f(x) = x\}$ is a continuous domain with respect to the induce order of $L$ and $\text{fix}(f)$ is the retract of $L$. In
general, \( \text{fix}(f) \) is not continuous. In the web page [1], Huth gave a continuous domain \( L \) and a Scott continuous self-mapping \( f : L \rightarrow L \) satisfying that \( \text{fix}(f) \) is not continuous. He also put forward the following problem:

**Problem** For which classes of continuous domains \( L \) are there Scott continuous self-maps \( f : L \rightarrow L \) such that \( \text{fix}(f) \) are not continuous on the induced order?

If \( L \) is an algebraic domain with a countable base and the smallest element, this problem is answered in this paper.

On the other hand, Kou [2] considered those continuous domains \( L \) and Scott continuous self-mappings \( f : L \rightarrow L \) such that \( \text{fix}(f) \) is continuous, and discussed whether \( \text{fix}(f) \) is the retract of \( L \) if \( \text{fix}(f) \) is continuous. Kou proved the following result.

**Theorem 1.1** ([2]) Suppose that \( L \) is a continuous \( L \)-domain and \( f : L \rightarrow L \) is a stable mapping. Then \( \text{fix}(f) \) is a continuous domain; if \( L \) is simultaneously a complete lattice, then \( \text{fix}(L) \) is the retract of \( L \).

For the case of \( L \)-domain, \( f \) is a stable mapping if and only if \( f \) preserves directed supremum and compatible infimum. Kou [2] constructed a continuous \( L \)-domain and a stable mapping \( f : L \rightarrow L \) such that \( \text{fix}(f) \) is not the retract of \( L \). In general, an \( L \)-domain is not necessarily a bounded complete domain. For example, \( L = \{\perp, a, b, c, d\} \) ordered by: \( \perp \) is the smallest element; \( a, b \) are incomparable; \( c, d \) are incomparable; \( a < c, d \) and \( b < c, d \). \( L \) is an \( L \)-domain, but \( L \) is not bounded complete since \( a, b \) have no supremum though \( a, b \) are bounded. For a bounded completely continuous domain \( L \) and stable mapping \( f : L \rightarrow L \), is \( \text{fix}(f) \) the retract of \( L \)? In this paper, an example is given and shows the answer is no.

## 2. Main results

Huth [1] gave a continuous domain \( D \) and a Scott continuous self-mapping \( f : D \rightarrow D \) such that \( \text{fix}(f) \) is not continuous. But his example is somewhat complex and refers to the other results. We give a simple example as follows.

**Example 2.1** Let \( D = \{a_0, a_1, a_2, \ldots, a_n, \ldots, a_\infty, b_0, b_1, b_2, \ldots, b_n, \ldots, b_\infty\} \) ordered by:

\[
\forall i \in \mathbb{N}, \ a_i < a_{i+1} < a_\infty, \ b_i < b_{i+1} < b_\infty, \ a_i < b_i, \ a_\infty < b_\infty.
\]

Then there is a Scott continuous self-mapping \( f : D \rightarrow D \) such that \( \text{fix}(f) \) is not continuous on the induced order of \( D \).

Define \( f : D \rightarrow D \) as follows:

\[
f(a_0) = a_0, \ f(a_\infty) = a_\infty, \ f(b_\infty) = b_\infty, \ f(a_i) = a_{i-1}, \ i \geq 1, \ f(b_i) = b_i, \ \forall \ i \in \mathbb{N}.
\]

Then \( \text{fix}(f) = \{a_0, a_\infty, b_0, b_1, \ldots, b_n, \ldots, b_\infty\} \). Obviously, \( \text{fix}(f) \) is not continuous on the induced order.

At the end of the paper [1], Huth proposed the following problem: For which classes of continuous domains \( D \) are there Scott continuous self-maps \( f : D \rightarrow D \) such that \( \text{fix}(f) \) is not
continuous on the induced order? The following theorem gives a partial answer.

**Theorem 2.2** Suppose that $D$ is an algebraic domain with a smallest element and a countable base. Then there is a Scott continuous self-map $f : D \to D$ such that $\text{fix}(f)$ is not continuous on the induced order if and only if there is $x, y \in D$ such that $x < y$, but $x \preceq_D y$ does not hold.

**Proof** We firstly show necessity. Suppose there is a Scott continuous self-map $f : D \to D$ such that $\text{fix}(f)$ is not continuous on the induced order. Since $\text{fix}(f)$ is not continuous, there is $x \in \text{fix}(f)$ such that $\text{fix}(f)$ is not continuous at $x$, therefore, $x \preceq_{\text{fix}(f)} x$ does not hold. Then there is a directed subset $\{x_i\}_{i \in I} \subseteq \text{fix}(f)$ such that $\bigvee_{i \in I} x_i \geq x$ and $\forall i \in I, x_i \not\geq x$. Let $y = \bigvee_{i \in I} x_i$. If $y > x$, then $x \preceq_{\text{fix}(f)} y$ does not hold. Otherwise, there is $i_0 \in I$ such that $x \leq x_{i_0}$, contradiction follows. So $x \preceq_D y$ does not hold. Since $x = y$, then there is $x_{i_0}$ such that $x_{i_0} < x$, but $x_{i_0} \preceq_D x$ does not hold. Otherwise, suppose for any $x_i, x_i \preceq_D x$. Since $\{x_i\}_{i \in I}$ is directed, $\text{fix}(f)$ is continuous at $x$. This contradicts the continuity of $\text{fix}(f)$ at $x$. Necessity is proved.

Then we show sufficiency. Suppose that there are $x, y \in D$ such that $x < y$, but $x \preceq y$ does not hold. Since $D$ has a countable base, there is a countable ascending chain $\{y_i\}_{i \in I}$ such that $\forall i \in I, y_i \in K(D), \bigvee_{i \in I} y_i = y$, but $\forall i \in I, x \not\leq y_i$. Similarly, there is a countable ascending chain $\{x_j\}_{j \in J}$ such that $\forall j \in J, x_j \in K(D), \bigvee_{j \in J} x_j = x$. Let $x_0 = 0 = \top, y_0 = b_0$. We can suppose that $b_0 \not\leq x$. Take $x_1$ such that $x_1 > \max\{x_j \mid x_j \leq y_0, j \in J\}$. Let $x_1 = a_1$. Since $a_1 < x < y$, there is $i_1 \in I$ such that $a_1 < y_i$. Take $x_{j_2}$ such that $x_{j_2} > \max\{x_j \mid x_j \leq y_{i_1}, j \in J\}$. Let $x_{j_2} = a_2, y_{i_1} = b_1$. Inductively, for any $n \in \mathbb{N}$, $a_n$ and $b_n$ are defined. Let $x = a_\infty, y = b_\infty$. We give a copy of the domain in Example 2.1. Denote this copy as $D_1$. We project $D$ onto $D_1$ and define a map $f_1 : D \to D_1$ as follows:

$$\forall x \in D, f_1(x) = \bigvee_{D_1} \{e \in C \mid e \leq x\},$$

in which $C = \{a_0, a_1, \ldots, a_n, \ldots, b_0, b_1, \ldots, b_n, \ldots\}$.

Let $h = f \circ f_1$, in which $f$ is the map in Example 2.1. Then $\text{fix}(h)$ is not continuous on the induced order. The proof is completed. $\square$

We give a bounded complete continuous domain $L$ and a stable map $f : L \to L$ such that $\text{fix}(f)$ is not the retract of $L$.

**Example 2.3** Let $L = \{a_0, a_1, \ldots, a_n, \ldots, a_\infty, b_1, b_2, \ldots, b_n, \ldots, b_\infty\}$ ordered by: $a_0$ is the smallest element; $\forall i \in \mathbb{N}, a_i < b_i, a_\infty < b_\infty, a_0 < a_1 < \cdots < a_n < \cdots < a_\infty$, where the order of $\{b_0, b_1, b_2, \ldots, b_n, b_\infty\}$ is discrete. Then $L$ is a bounded complete domain. We give a stable map $f : L \to L$ such that $\text{fix}(f)$ is not the retract of $L$.

Define $f : L \to L$ as follows:

$$\forall i \in \mathbb{N}, f(b_i) = b_i, f(a_{i+1}) = a_i, f(b_\infty) = b_\infty, f(a_\infty) = a_\infty, f(a_0) = a_0.$$

Then $\text{fix}(f) = \{a_0, b_0, b_1, b_2, \ldots, b_n, \ldots, b_\infty, a_\infty\}$. Next we show that $\text{fix}(f)$ is not the retract of $L$. Suppose that $\text{fix}(f)$ is the retract of $L$. Then there is a Scott continuous map $p : L \to \text{fix}(f)$
and \( r : \text{fix}(f) \to L \) such that \( p \circ r = \text{id}_{\text{fix}(f)} \).

We can claim \( \forall i \in \mathbb{N}, p(a_i) = a_0 \). In fact, suppose there is \( i_0 \in \mathbb{N} \) such that \( p(a_{i_0}) \neq a_0 \). Then there is \( n_0 \in \mathbb{N} \) such that \( p(a_{i_0}) = b_{n_0} \) or \( b_\infty \) or \( a_\infty \). Suppose that \( p(a_{i_0}) = b_{n_0} \). Then \( \forall i \in \mathbb{N} \) and \( i \geq i_0, p(b_i) \geq p(a_{i_0}) = b_{n_0}, p(a_i) \geq p(a_{i_0}) = b_{n_0} \), therefore, \( p(b_i) = b_{n_0}, p(a_i) = b_{n_0} \). It implies that \( \text{im}(p) \) is finite and contradicts \( p \circ r = \text{id}_{\text{fix}(f)} \). Similarly, suppose \( p(a_{i_0}) = b_\infty \) or \( a_\infty \). Then \( \text{im}(p) \) is finite. This contradicts \( p \circ r = \text{id}_{\text{fix}(f)} \). Hence \( \forall i \in \mathbb{N}, p(a_i) = a_0 \) and by the continuity of \( p \), \( p(a_\infty) = a_0 \).

Consider the map \( r : \text{fix}(f) \to L \). We can claim \( r(a_\infty) \neq a_\infty \). Otherwise, \( p \circ r(a_\infty) = p(a_\infty) = a_0 \). This contradicts \( p \circ r = \text{id}_{\text{fix}(f)} \). Similarly, \( \forall i \in \mathbb{N}, r(a_\infty) \neq a_i \), therefore, \( r(a_\infty) \in \{b_0, b_1, \ldots, b_n, \ldots, b_\infty\}, r(a_\infty) \neq b_\infty \). Otherwise, \( r(a_\infty) = r(b_\infty) = b_\infty \). This contradicts \( p \circ r = \text{id}_{\text{fix}(f)} \). Similarly, \( \forall i \in \mathbb{N}, r(a_\infty) \neq b_i \). Such \( r \) does not exist. \( \text{fix}(f) \) is not the retract of \( L \). The proof is completed. \( \Box \)

If the stable map is strengthened to preserve arbitrary infimum, then we have the following result. The proof is basically the same as that in [2, Theorem 2.2].

**Theorem 2.4** Suppose that \( L \) is a bounded complete domain, \( f : L \to L \) is Scott continuous and preserves arbitrary infimum and \( \forall x \in L, \text{fix}(f) \cap \uparrow x \neq \emptyset \). Then \( \text{fix}(f) \) is the retract of \( L \).

**Proof** Define a map \( r : L \to \text{fix}(f) \) as follows: \( \forall x \in L \)

\[
r(x) = \land (\uparrow x \cap \text{fix}(f))
\]

in which the infimum is defined on \( L \). Since \( f \) preserves arbitrary infimum, \( \forall x \in L, \land (\uparrow x \cap \text{fix}(f)) = f(\land (\uparrow x \cap \text{fix}(f))) \in \text{fix}(f) \). So \( r \) is well defined and preserves the order. Next, we show \( r \) is Scott continuous.

Suppose that \( D \subseteq L \) is directed and \( y = \lor D, \forall d \in D, d \leq \land (\uparrow d \cap \text{fix}(f)) = r(d) \), therefore

\[
y = \lor D r(d).
\]

Since \( r(y) = \land (\uparrow y \cap \text{fix}(f)) \) and \( \lor_d D r(d) \in \text{fix}(f), r(y) = r(\lor D) = \lor _d D r(d). \)

Let \( s : \text{fix}(f) \to L \) be inclusion map. Then \( r \circ s = \text{id}_{\text{fix}(f)} \). The proof is completed. \( \Box \)

**Remark** The map in Example 2.3 does not preserve arbitrary infimum.

**Acknowledgement** The authors would like to thank Martin HUTH and Hui KOU for the helpful discussions with them about the problem of fixed points of Scott continuous self-map.

**References**


