Self-reciprocal Functions and Self-reciprocal Transforms*

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The principal object of this paper is to show that certain general classes of self-reciprocal series transforms can be constructed by making use of analytic functions $\phi(z)$ such that $\phi[\phi(z)]=z$. A brief analysis has been given of a kind of self-reciprocal integral transforms by the aid of Widder's inversion theory for the Laplace-Lebesgue integral.

Introduction

By the so-called "self-reciprocal series transform" we mean a pair of self-reciprocal formulas of the form

$$f(n)=\sum_k S(n,k)g(k) \text{ and } g(n)=\sum_k S(n,k)f(k)$$

where $S(n,k)$ is called the kernel of series transformation, and $\{f(k)\}$ and $\{g(k)\}$ are any two sequences of real or complex numbers connected by the inverse series relations. It was found as early as in 1964 that certain classes of self-reciprocal seies transforms as well as integral transforms can be constructed by means of self-reciprocal analytic functinns (cf. [1],[2],[3],[4]). In this paper we shall present a full development of the general technique proposed previously. A number of general theorems (a few of which were announced earlier) will be proved and discussed in some detail.

§1 Real and Complex Self-reciprocal Functions

The term "self-reciprocal function" has been commonly adopted in the theory of Fourier transforms. Here we shall employ the term with different meaning. Roughly every function with its inverse equal to itself may be called a self-reciprocal function. Precise definitions are given as follows.

* 1981年4月29日收到.

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Definition 1 (Real case). Let $\lambda>0$. We shall say that $y=\phi(x)$ is a real valued self-reciprocal function defined on $[0,\lambda]$ whenever the functional relation $\phi(y)=\phi[\phi(x)]=x$ is valid for $0\leqslant x\leqslant\lambda$, $0\leqslant y\leqslant\lambda$ with $\phi(0)=\lambda$, $\phi(\lambda)=0$.

Definition 2 (Complex case). An analytic function $w=\phi(z)$ is said to be self-reciprocal in a connected domain $\mathscr{D}$ in the complex z-plane whenever the function $\phi(w)$ is analytic for all the values $w=\phi(z)$ with $z\in\mathscr{D}$, and the reciprocity relation $\phi[\phi(z)]=z$ holds throughout $\mathscr{D}$.

Evidently, for the real case the function graph of $y=\phi(x)(0\leqslant x\leqslant\lambda)$ consists of all those pairs of points $(x,y)\equiv[x,\phi(x)]$ and $[y,\phi(y)]\equiv(y,z)$ that are symmetrical with respect to the line of symmetry $y=x$. This characteristic of the graph suggests that one can exhibit or construct as many self-reciprocal functions as one likes.

Note that in Definition 2 it is unnecessary to assume that $\phi(z)\in\mathscr{D}$ for $z\in\mathscr{D}$. However, the functional relation $\phi[\phi(z)]=z$ should imply that the relation between z and $w=\phi(z)$ is that of a one-one correspondence. In particular, if the reciprocity relation $\phi[\phi(z)]=z$ is valid in such a domain $\mathscr{D}$ that $\phi(z)\in\mathscr{D}$ for $z\in\mathscr{D}$, then $\mathscr{D}$ is called the domain of self-reciprocity for the self-reciprocal function $\phi(z)$. Of course one can still consider the largest possible domain of self-reciprocity for $\phi(z)$. However, for reciprocal series transforms we need only the local formulation of the self-reciprocity of $\phi(z)$. As a matter of fact, in establishing certain general theorems for reciprocal series transforms one often requires only the analytic self-reciprocity of $\phi(z)$ in a certain neighborhood of $z=0$,

Generally one can always find various self-reciprocal functions $z'=\phi(z)$ by solving various symmetrical equations of the form

$$F(z,z')\equiv F(z',z)=0.$$

Let us state here a few simple examples to illustrate this point.

Example 1. Consider the real symmetric equation

$$F(x,x')\equiv 4-e^{x}-e^{x'}=0, \quad (0\leqslant x,x'\leqslant\log 3).$$

Solving, we get the self-reciprocal function

$$x'=\phi(x)=\log(4-e^{x}).$$

It is clear that $\phi(x)$ decreases monotonically from log3 to 0 as x increases from 0 to log3. Hence $[0,\log 3]$ may be taken as the valid interval for the function obtained.

Example 2. Consider the complex-valued symmetric equation

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$$F(z,z')\equiv zz'-\alpha(z+z')-\beta=0,$$

where α and β are any two real or complex constants with $\alpha\neq0$ and $\beta\neq-\alpha^2$. Solving the equation, we obtain

$$z'=\phi(z)=(\alpha z+\beta)/(z-\alpha).$$

This is a self-reciprocal function of z for $|z|<\rho\leqslant|\alpha|$, ρ being a fixed positive number. In fact, it may be verified directly that $\phi[\phi(z)]=z$ holds in the open disk $\mathscr{D}$ $(|z|<\rho)$.

In particular, taking $\alpha=1$ and $\beta=0$, we obtain the simple self-reciprocal function $\phi(z)=z/(z-1)$ in $\mathscr{D}(|z|<1)$. As may be verified at once, this function has a domain of self-reciprocity containing a neighborhood of the point $z=0$ in the complex plane.

It may be worthy of mention that one can always start from a given self-reciprocal function $\phi(z)$ to generate a variety of self-reciprocal functions by means of suitable substitutions of variables. A few immediate observations may be made as follows:

1° If $\phi(z)$ is a self-reciprocal function obtainable from a given symmetrical equation, then so is $\psi(z)=[\phi(\alpha z+\beta)-\beta]/\alpha$ with $\alpha\neq0$, α and β being real or complex constants. In fact, supposing $z'=\phi(z)$ come from an equation $F(z,z')\equiv F(z',z)=0$, we see that

$$F(\alpha z+\beta,\quad \alpha z'+\beta)\equiv F(\alpha z'+\beta,\quad \alpha z+\beta)=0$$

is also a symmetrical equation, which gives at once the self-reciprocal function $z'=[\phi(\alpha z+\beta)-\beta]/\alpha$. Sometimes, the particular case $w=\phi(\alpha z)/\alpha$ $(\alpha\neq0)$ may be found useful.

2° Generally, if θ denotes a one-to-one (analytic) mapping with θ^{-1} as its inverse, then it follows from the symmetrical equation

$$F[\theta(z),\quad \theta(z')]\equiv F[\theta(z'),\quad \theta(z)]=0$$

that $z'=\theta^{-1}\phi[\theta(z)]$ is also a self-reciprocal function. This general substitution rule applies to transforming various self-reciprocal functions.

§2 Self-reciprocal Transforms Constructed by Analytic Self-reciprocal Functions

In this section we shall expound what a kind of self-reciprocal series transforms can be built up by analytic self-reciprocal functions. The following rasult was announced in an earlier abstract [2] without proof.

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Theorem 1. Let $w=\phi(z)$ be an analytic function self-reciprocal in the open disk $|z|<\rho(\rho\leqslant 1)$ with $|\phi(0)|<\rho$ and $|\phi(z)|<1$. Then we have the following pair of self-reciprocal formulas

$$n!\,g(n)=\sum_{k=0}^{\infty}f(k)D_z^n[\phi(z)]^k|_{z=0}, \tag{1}$$

$$n!\,f(n)=\sum_{k=0}^{\infty}g(k)D_z^n[\phi(z)]^k|_{z=0}, \tag{2}$$

provided that either $\sum_0^\infty f(k)$ or $\sum_0^\infty g(k)$ is convergent, where $D_z^n=(d/dz)^n$ $(n=0,1,2,\cdots)$.

Proof. Suppose that $\sum_0^\infty f(k)$ is convergent. Let us consider the series

$$G(z)=\sum_{k=0}^{\infty}f(k)[\phi(z)]^k,\quad (|z|<\rho).$$

It is readily shown that $G(z)$ is uniformly convergent along every circle $|z|=\rho_1<\rho$. Indeed, the condition $|\phi(z)|<1(|z|<\rho)$ implies that

$$\max_{|z|=\rho_1}|\phi(z)|<1,\ \lim_{k\to\infty}[\phi(z)]^k=0\quad \text{for } |z|=\rho_1.$$

Consequently we have

$$\sum_{k=0}^{\infty}|(\phi(z))^k-(\phi(z))^{k+1}|=\sum_{k=0}^{\infty}|\phi(z)|^k\cdot|1-\phi(z)|\leqslant 2\sum_{k=0}^{\infty}|\phi(z)|^k<\text{const.}$$
$$(|z|=\rho_1).$$

Thus Abel's test affirms that the series $G(z)$ converges uniformly for $|z|=\rho_1$;

Note that for each $k(k=1,2,\cdots)$ the function $(\phi(z))^k$ is analytic in $|z|<\rho$, so that we have the convergent power series expansion

$$(\phi(z))^k=\sum_{n=0}^{0}\frac{1}{n!}[D_z^n(\phi(z))^k]_{z=0}\cdot z^n,\quad (k=1,2,\cdots).$$

Thus the condition of the well-known Weierstrass double series theorem (cf. Weierstrass [6] or T. J. Bromwich, Theory of Infinite Series, p. 266) are all fulfilled for $G(z)$, and consequently we obtain a power series expansion as follows

$$G(z)\equiv\sum_{k=0}^{\infty}f(k)\,(\phi(z))^k=\sum_{n=0}^{\infty}g(n)\cdot z^n, \tag{3}$$

where $|z|<\rho$, and $g(n)$ $(n=0,1,2,\cdots)$ are given by

$$g(n)=\frac{1}{n!}\sum_{k=0}^{\infty}f(k)D_z^n(\phi(z))^k|_{z=0}. \tag{1 bis}$$

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Now making use of the self-reciprocity of the function $w=\phi(z)$ we have $z=\phi(w)$ with $w\in U$, where U (in the complex w-plane) is the image of the open disk $|z|>\rho$ (in the complex z-plane). Clearly U is included in the unit circle $|w|<1$ and contains the origin $w=0$ as its interior point inasmuch as $|\phi(0)|<\rho$ and $\phi(\phi(0))=0$.

Let us take in the w-plane a sufficiently small closed disk with center at $w=0$, say V ($|w|\leqslant\delta$) such that $V\subset U$. Obviously, the image of V, denoted by $\phi(V)$, will be entirely included in the open disk $|z|<\rho$. Thus if we rewrite (3) in the form

$$F(w)\equiv\sum_{k=0}^{\infty}g(k)(\phi(w))^k=\sum_{n=0}^{\infty}f(n)w^n, \tag{4}$$

we may infer that the series $F(w)$ on the left-side of (4) is also uniformly convergent on V ($|w|\leqslant\delta$). And consequently the Weierstrass double series theorem does also apply to the double series $F(w)$ (when each $(\phi(w))^k$ has been expanded in powers of w with $|w|<\delta$), so that we obtain in a similar manner

$$f(n)=\frac{1}{n!}\sum_{k=0}^{\infty}g(k)D_w^n(\phi(w))^k|_{w=0}. \tag{2bis}$$

This completes the proof.

Remark The uniform convergence of the series $G(z)$ may be verified by the aid of Abel-Dirichlet's test. Thus the convergence condition imposed upon $\sum_0^{\infty}f(k)$ (or $\sum_0^{\infty}g(k)$) may be replaced by the following assumption

$$\sum_0^{\infty}|f(k)-f(k+1)|<A,\quad \overline{\lim_{k\to\infty}}|f(k)|<A,$$

A being a positive constant.

A particular case of Theorem 1 seems to be interesting, and we may state it as a corollary.

Corollary 1. For every analytic self-reciprocal function $\phi(z)$ with $\phi(0)=0$, we have a pair of reciprocal relations as follows

$$n!\,g(n)=\sum_{k=1}^{n}f(k)D_z^n(\phi(z))^k|_{z=0}, \tag{1'}$$

$$n!\,f(n)=\sum_{k=1}^{n}g(k)D_z^n(\phi(z))^k|_{z=0}. \tag{2'}$$

Correspondingly we have the orthogonality relation for $n\geqslant m\geqslant1$,

$$\sum_{k=m}^{n}\frac{D^n(\phi(0))^k}{n!}\cdot\frac{D^k(\phi(0))^m}{k!}=\delta_{n,m},$$

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where $D^n(\phi(0))^k = D_z^n(\phi(z))^k_{z=0}$ and $\delta_{n,m}$ is the Kronecker delta.

In fact, the condition $\phi(0)=0$ implies that the self-reciprocal function $w=\phi(z)$ should take the form $\phi(z)=z+c_1z^2+\cdots$ in a neighborhood of $z=0$, so that we have

$$D_z^n(\phi(z))^k|_{z=0}=0, \quad (k=n+1,n+2,\cdots).$$

It may be observed that the condition $|\phi(z)|<1$ involved in the hypothesis of Theorem 1 may be removed. Actually we have the following theorem as an immediate extension of Theorem 1.

Theorem 2: Let $\phi(z)$ be an analytic self-reciprocal function defined in the open disk $|z|<r$ with $|\phi(0)|<r$. Let M be a positive constant such that

$$M\geqslant r, \quad M>\sup_{|z|<r}|\phi(z)|,$$

and let $\phi(z)$ remain analytic for $|z|<M$. Then for the self-reciprocal function $\psi(z)=\phi(Mz)/M$ we have the inverse series relations

$$n!\,g(n)=\sum_{k=0}^{\infty}f(k)D_z^n(\psi(z))^k|_{z=0}, \tag{5}$$

$$n!\,f(n)=\sum_{k=0}^{\infty}g(k)D_z^n(\psi(z))^k|_{z=0}, \tag{6}$$

provided that at least one of the series $\sum_0^\infty f(k)$ and $\sum_0^\infty g(k)$ is convergent.

Proof. We have assumed that $\psi(z)$ is analytic for $|z|<1$. Now denote $\rho=r/M$ so that $\rho\leqslant1$. By the maximum-modulus theorem we have for $|z|<\rho$,

$$|\psi(z)|=|\phi(Mz|/M<\sup_{|z|<\rho}|\phi(Mz)|/M=\sup_{|z|<r}|\phi(z)|/M<1.$$

Moreover, we have $|\psi(0)|=|\phi(0)|/M<\rho$. Hence Theorem 2 follows from Theorem 1 with $\phi(z)$ being replaced by $\psi(z)$.

Theorems 1 and 2 can be used as a tool for finding various special pairs of self-reciprocal formulas. Let us now give a few simple examples as follows.

Example 1. Taking the simple self-reciprocal function $\phi(z)=z/(z-1)$, $(z\neq1)$, we have $\phi(0)=0$ and

$$\frac{1}{n!}\left(\frac{d}{dz}\right)^n[z^k(z-1)^{-k}]_{z=0}=(-1)^k\binom{n-1}{k-1}, \quad (k=1,2,\cdots).$$

Thus, applying (1) and (2) we find the pair of simple inversion formulas

$$g(n)=\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}f(k),$$

$$f(n)=\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}g(k).$$

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This is the well-known pair for δ-transform (cf. G. H. Hardy [5], Theorem 196)

Example 2. Taking $\phi(z)=\frac{1}{3}-z$ for $|z|<\frac{1}{2}$, we have

$$|\phi(0)|=\frac{1}{3}<\frac{1}{2},\quad |\phi(z)|\leqslant\frac{5}{6}<1.$$

Hence we obtain by means of Theorem 1 with $\rho=\frac{1}{2}$ the following reciprocal pair

$$3^{-n}g(n)=(-1)^n\sum_{k=0}^{\infty}\binom{k}{n}3^{-k}f(k),$$

$$3^{-n}f(n)=(-1)^n\sum_{k=0}^{\infty}\binom{k}{n}3^{-k}g(k),$$

whenever $\sum_0^\infty f(k)$ is convergent. This is known as the pair for δ^*-transform (also called Bernoulli-Euler's transform).

Example 3. Let α and β be real parameters such that $\alpha>0$, $0\leqslant\beta<1/(2+\alpha)$. Define $G(n,k;\alpha,\beta)$ by the equation

$$G(n,k;\alpha,\beta)=\sum_{v=0}^{k}\binom{k}{v}\binom{n+k-v-1}{k-1}\alpha^{n-v}\beta^{k-v}.$$

Then we have the self-reciprocal pair

$$g(n)=\sum_{k=0}^{\infty}(-1)^kG(n,k;\alpha,\beta)f(k),\tag{7}$$

$$f(n)=\sum_{k=0}^{\infty}(-1)^kG(n,k;\alpha,\beta)g(k),\tag{8}$$

provided that either $\sum_0^\infty f(k)$ or $\sum_0^\infty g(k)$ is convergent.

In this example we have to make the usual conventions for binomial coefficients:

$$\binom{n-1}{-1}=0\ (n\neq0),\quad \binom{-1}{-1}=1,\quad \binom{-1}{k-1}=0\ (k\neq0).$$

Actually this example follows from using the self-reciprocal function

$$w=\phi(z)=(z+\beta)/(\alpha z-1)$$

defined in the open disk $|z|<\rho=(1-\beta)/(1+\alpha)$. The conditions of Theorem 1 may be verified for $|z|<\rho$.

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$$|\phi(z)|\leqslant\frac{|z|+\beta}{||\alpha z|-1|}<\frac{\rho+\beta}{1-\alpha\rho}=\frac{1+\alpha\beta}{1+\alpha\beta}=1,$$

$$|\phi(0)|=\rho<\frac{1}{2+\alpha}=\frac{1-1/(2+\alpha)}{1+\alpha}<\frac{1-\beta}{1+\alpha}=\rho.$$

Moreover, we easily find

$$D_z^n\left(\frac{z+\beta}{\alpha z-1}\right)^k\Big|_{z=0}=(-1)^k\sum_{v=0}^{k}\binom{k}{v}\binom{n+k-v-1}{k-1}\alpha^{n-v}\beta^{k-v}.$$

This is precisely $(-1)^k\ G(n,k;\alpha,\beta)$ and therefore (7) and (8) follow as a consequence of Theorem 1.

Theorem 1 affirms that the reciprocal formulas (1) and (2) can be deduced from the reciprocity relation $w=\phi(z)\Longleftrightarrow z=\phi(w)$ (i. e. $\phi(\phi(z))=z$). However we have not yet known whether the converse proposition is true or not. The following theorem may be regarded as having offered a partial answer to the question just mentioned.

Theorem 3. Let $w=\phi(z)$ be a function analytic in the open disk $|z|>\rho$ $(\rho\leqslant 1)$ with $|\phi(0)|<\rho$ and $|\phi(z)|<1$. Suppose that the reciprocal relations (1) and (2) hold for all the ralated sequences $\{f(k)\}$ and $\{g(k)\}$ with both $\sum_0^\infty f(k)$ and $\sum_0^\infty g(k)$ being convergent. Then the function $\phi(z)$ must be self-reciprocal in a certain neighborhood of $z=0$.

Proof. In accordance with the proof of Theorem 1 we may write

$$G(z)\equiv\sum_{k=0}^{\infty}f(k)w^k=\sum_{n=0}^{\infty}g(n)\cdot z^n, \qquad (3\text{ bis})$$

where $|z|<\rho$, $w=\phi(z)$ and $g(n)$ is precisely given by (1). Notice that the assumption $(1)\Longleftrightarrow(2)$ implies that the related sequences $\{f(n)\}$ and $\{g(n)\}$ contained in (3 bis) can be exchanged in their places. Thus (3 bis) may be changed into the form

$$\sum_{k=0}^{\infty}g(k)w^k=\sum_{n=0}^{\infty}f(n)z^n,$$

where $w=\phi(z)$. But now this functional relation between two power series should imply the relation $z=\phi(w)$ $(|w|<\rho)$ in accordance with (3 bis). Hence the reciprocity $z=\phi(\phi(z))$ holds for z with $|\phi(z)|<\rho$, and the theorem is proved.

As shown in the above theorem, the convergence condition imposed upon both $\sum_0^\infty f(k)$ and $\sum_0^\infty g(k)$ seems to be too stringent. However, whether such a condition can be lightened or not is not known.

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In establishing a result parallel with Theorem 1 we may assume that $w=\phi(z)$ is not self-reciprocal. Certainly in such a case one can only get a pair of reciprocal (but not self-reciprocal) series relations. More precisely we have the following

Theorem 4. Let $w=\phi(z)$ be a simple (schlicht) function defined for $|z|<\rho$ $(\rho\leqslant 1)$ with $|\phi(0)|<\rho$ and $|\phi(z)|<1$. Then for every sequence $\{f(k)\}$ such that $\sum_0^\infty f(k)$ is convergent, we have the following pair or reciprocal formulas

$$n!g(n)=\sum_{k=0}^{\infty}f(k)D_z^n(\phi(z))^k|_{z=0}, \tag{9}$$

$$n!f(n)=\sum_{k=0}^{\infty}g(k)D_z^n(\psi(z))^k|_{z=0}, \tag{10}$$

provided that $z=\psi(w)$ is the inverse function of $w=\phi(z)$ with $|\psi(0)|<\rho$.

This theorem can be proved exactly in the same manner as that of Theorem 1. In fact, only the second part of the original proof has to be modified in such a way that the inverse function should take the form $z=\psi(w)$ instead of $z=\phi(w)$ and that (2 bis) be replaced by (10).

As a particular case of Theorem 4 we have the following

Corollary 2. Suppose that the simple function $w=\phi(z)$ has the property $\phi(0)=0$. Then (9) and (10) may be rewritten in the form

$$n!g(n)=\sum_{k=1}^{n}f(k)D_z^n(\phi(z))^k|_{z=0}, \tag{9'}$$

$$n!f(n)=\sum_{k=1}^{n}g(k)D_z^n(\psi(z))^k|_{z=0}, \tag{10'}$$

where $n=1,2,3,\cdots$

Example 4. Taking $\phi(z)=\sin z$ and $\phi(z)=\log(1+z)$ we get at once the following two reciprocal pairs

$$\begin{cases} n!g(n)=\sum_{k=1}^{n}f(k)D_x^n(\sin x)^k|_{x=0}, \\ n!f(n)=\sum_{k=1}^{n}g(k)D_x^n(\arcsin n)^k|_{x=0}, \end{cases}$$

and

$$\begin{cases} n!g(n)=\sum_{k=1}^{n}f(k)D_x^n(\log(1+x))^k|_{x=0}, \\ n!f(n)=\sum_{k=1}^{n}g(k)D_x^n(c^x-1)^k|_{x=0}. \end{cases}$$

Here it is no real restriction to assume $\phi(x)'s$ to be real functions of a real variable;

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§3 Domain of Self-reciprocity for Analytic Self-reciprocal Functions

Since the concept "domain of self-reciprocity" is so important for our method of construction of self-reciprocal transforms, we give here a brief account of it.

Let $\mathfrak{S}$ be a simply connected domain in the complex plane, and let $\phi(z)$ be a analytic function self-reciprocal in $\mathfrak{S}$, i. e. $\phi(\phi(z))=z$ for $z\in\mathfrak{S}$. Denote by

$$\mathfrak{S}^*=\{z' \mid z'=\phi(z), z\in\mathfrak{S}\}$$

the set of images of the function $\phi(z)$, or in abbreviation, we may write $\mathfrak{S}^*=\phi(\mathfrak{S})$. Then the intersection set $\mathscr{D}=\mathfrak{S}\cap\mathfrak{S}^*$, if non-void, will be called a (local) domain of self-reciprocity for $\phi(z)$. It is easy to prove the following

Proposition 1. For $\mathscr{D}=\mathfrak{S}\cap\mathfrak{S}^*$ with $\mathfrak{S}^*=\phi(\mathfrak{S})$ we have the identity $\phi(\mathscr{D})=\mathscr{D}$.

Proof. By definition we have $\mathscr{D}\subset\mathfrak{S}$ and $\mathscr{D}\subset\mathfrak{S}^*$. Thus it follows that

$$\phi(\mathscr{D})\subset\phi(\mathfrak{S})=\mathfrak{S}^*,\quad \phi(\mathscr{D})\subset\phi(\mathfrak{S}^*)=\mathfrak{S}.$$

Consequently we have

$$\phi(\mathscr{D})\subset\mathfrak{S}^*\cap\mathfrak{S}=\mathscr{D},$$

$$\phi(\phi(\mathscr{D}))\subset\phi(\mathscr{D})\subset\mathscr{D}.$$

But $\phi(\phi(\mathscr{D}))=\mathscr{D}$. Hence we may conclude that $\phi(\mathscr{D})=\mathscr{D}$.

From the proposition just proved we may infer that every self-reciprocal function $\phi(z)$ can be trausformed into a new self-reciprocal function $\psi(z)$ such that $\psi(0)=0$. As a matter of fact, the well-known fixed point principle applies to the mapping ϕ: $\phi(\mathscr{D})=\mathscr{D}$, so that there is a point $\xi\in\mathscr{D}$ for which $\phi(\xi)=\xi$. Let us now define

$$\psi(z)=(\phi(az+\xi)-\xi)a,\quad (a\neq 0).$$

Evidently $\psi(z)$ is self-reciprocal with $\psi(0)=0$.

The so-called "complete domain of self reciprocity" for $\phi(z)$ may be defined as the largest possible domain $\mathscr{D}$ such that $\phi(\mathscr{D})=\mathscr{D}$. A more general definition may be described as follows.

Given an analytic function $z'=\phi(z)$, two points z_1 and z_2 in the complex plane are said to be a pair of reciprocal points of $\phi(z)$, if $\phi(z)$ is analytic at these points with $z_1=\phi(z_2)$ and $z_2=\phi(z_1)$. Then the point set (non-empty set)

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$\mathscr{D}$ consisting of all the reciprocal points of $\phi(z)$ is called the complete domain of self-reciprocity for $\phi(z)$.

Evidently in the above formulation for the complete domain $\mathscr{D}$ the correspondence between $z\in\mathscr{D}$ and $z'\in\mathscr{D}$ must be one-to-one. The following simple proposition may be inferred immediately from the definition given above.

Proposition 2. The complete domain of self-reciprocity for an analytic self-reciprocal function $\phi(z)$ is the domain of analyticity of $\phi(z)$ in the whole complex z-plane, i. e. the z-plane without the singular points of $\phi(z)$.

In what follows we give a few simple examples.

Example 1. The self-reciprocal function $\phi(z)=-z$ takes the whole z-plane as its complete domain of self-reciprocity.

Example 2. The self-reciprocal function $\phi(z)=(\alpha z+\beta)/(z-\alpha)$ (α, β being complex constants with $\beta\neq-\alpha^2$) has the whole z-plane without the point $z=\alpha$ as its complete domain of self-reciprocity.

Example 3. If in the self-reciprocal function $w=z/(z-1)$ $(z\neq1)$ we replace z and w by $\sqrt{z}$ and $\sqrt{w}$ respectively, then we get another self-reciprocal function

$$w=\phi(z)=\left(\frac{\sqrt{z}}{\sqrt{z}-1}\right)^2=\frac{z}{(\sqrt{z}-1)^2},\quad (z\neq1).$$

Here we have to assume that both $\sqrt{z}$ and $\sqrt{w}$ (when inverted) take their principal values, (i. e. z and w are lying on the first sheet of the two-sheeted Riemann surfaces respectively), so that the monodromy of each of the functions $w=\phi(z)$ and $z=\phi(w)$ is still ensured. Evidently the self-reciprocal function so constructed has the whole z-plane (or the first sheet of the Riemann surface) without the singularity $z=1$ as its complete domain of self-reciprocity.

As regards the case $\phi(z)$ being a many-valued function, the condition $\phi(\mathscr{D})=\mathscr{D}$ for the domain of self-reciprocity of $\phi(z)$ should be in general replaced by the weaker one, viz. $\phi(\mathscr{D})\subset\mathscr{D}$. However, since the definition of the inverse function $z=\phi(w)$ $(w\in\phi(\mathscr{D}))$ can always be extended to the domain $\mathscr{D}$ by analytic continuation, we can still regard $\mathscr{D}$ as the domain of self-reciprocity for $\phi(z)$, Having adopted this convention, Proposition 2 may be generalized to the case where $\phi(z)$ is any many-valued analytic function defined on the z-plane.

Example 4. Consider the self-reciprocal function

$$w=\phi(z)=\log(4-e^z)$$

in the z-plane without the point $z=\log 4$. Obviously, this function can be made

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one-valued on the Riemann surface with infinitely many sheets joining the branch point $z=\log 4$. If we take the first sheet of the surface (or the z-plane without $z=\log 4$) as $\mathscr{D}$, the set of images, $\phi(\mathscr{D})$, is only a sub-set of $\mathscr{D}$, i. e. $\phi(\mathscr{D})\subset\mathscr{D}$. However, the domain of definition of $z=\log(4-e^w)$ can be readily extended to the whole domain $\mathscr{D}$. Thus the complete domain of self-reciprocity for $\phi(z)$ is precisely the whole complex plane without $z=\log 4$.

§4 Another Kind of Self-reciprocal Series Transforms

As having been noticed in §2, the condition $\phi(0)=0$ imposed upon the self-reciprocal function $\phi(z)$ may simplify the matter considerably. This is also the case for the matter presented in our 1964 paper [1]. In fact, the principal result contained in [1] can be simplified and refined to the following form:

Theorem 5. Let $\phi(z)$ be an analytic self-reciprocal function with $\phi(0)=0$, where $z=0$ is a simple zero of $\phi(z)$ contained in a local domain of self-reciprocity for $\phi(z)$. Then we have the pair of self-reciprocal formulas

$$n!g(n)=\sum_{k=1}^{n}kf(k)D_z^{n-1}\left\{z^{k-1}\left(\frac{z}{\phi(z)}\right)^n\right\}\Big|_{z=0},\tag{11}$$

$$n!f(n)=\sum_{k=1}^{n}kg(k)D_z^{n-1}\left\{z^{k-1}\left(\frac{z}{\phi(z)}\right)^n\right\}\Big|_{z=0},\tag{12}$$

where $n=1,2,3,\cdots$.

As had been shown loc. cit., the reciprocal relations (11)$\Longleftrightarrow$(12) can be established by making use of the well-known theorem of Bürmann and Teixeira in the theory of analytic functions (cf. Whittaker and Watson [7], ch. 7, §7. 31). Clearly, Theorem 5 can also be employed as a tool to find a variety of special inverse series relations.

§5 Self-reciprocal Transforms Constructed by Real-valued Self-reciprocal Functions

The kernels of transformation considered in the preceding sections take the forms

$$S(n,k)=\frac{1}{n!}D_z^n(\phi(z))^k_{z=0},\qquad S(n,k)=\frac{1}{n!}D_z^{n-1}\left\{z^{k-1}\left(\frac{z}{\phi(z)}\right)^n\right\}_{z=0}.$$

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These are built up by using differential operators, so that the self-reciprocal function $\phi(z)$ should be analytic. Is it possible to construct some kinds of kernels using integral operators instead of differential operators? The answer to this question is "yes".

As a matter of fact, if we are using Fourier expansion instead of Taylor's expansion as employed in the proof of Theorem 1, we shall find a new type of kernels as follows:

$$\Phi_c(n,k) = \frac{2}{\pi}\int_0^{\pi}\cos(nx)\cos(k\phi(x))\,dx, \tag{13}$$

$$\Phi_s(n,k) = \frac{2}{\pi}\int_0^{\pi}\sin(nx)\sin(k\phi(x))\,dx, \tag{14}$$

where $n,k=1,2,3,\cdots$, and $y=\phi(x)$ is a real self-reciprocal function defined on $0\leqslant x\leqslant\pi$ so that $\phi(0)=\pi$ and $\phi(\pi)=0$. Consequently we shall obtain two pairs of self-reciprocal formulas of the forms

$$g(n) = \sum_{k=1}^{\infty} f(k)\Phi_c(n,k), \tag{15}$$

$$f(n) = \sum_{k=1}^{\infty} g(k)\Phi_c(n,k); \tag{16}$$

and

$$g(n) = \sum_{k=1}^{\infty} f(k)\Phi_s(n,k), \tag{17}$$

$$f(n) = \sum_{k=1}^{\infty} g(k)\Phi_s(n,k); \tag{18}$$

Let us now establish the following

Theorem 6. Let $y=\phi(x)$ be a real self-reciprocal function defined on $0\leqslant x\leqslant\pi$, and let $\phi''(x)$ be continuous there. Then we have the reciprocal formulas (15) and (16), provided that both series $\sum_1^{\infty} f(k)k^2$ and $\sum_1^{\infty} g(k)k^2$ are absolutely convergent.

Proof. The function $\phi''(x)$ continuous throughout $[0,\pi]$, so is the function $\left(\frac{d}{dx}\right)^2\cos(k\phi(x))$ for every $k\,(k=1,2,\cdots)$. Consequently $\cos(k\phi(x)$ can be expanded into a convergent Fourier series

$$\cos(k\phi(x)) = \frac{1}{2}b_{k0} + \sum_{n=1}^{\infty} b_{kn}\cos(nx) \tag{19}$$

which converges absolutely and uniformly on $(0,\pi)$. Here the Fourier coefficients b_{kn} as given by

$$b_{kn} = \frac{2}{\pi}\int_0^{\pi}\cos(k\phi(x))\cos(nx)\,dx, \quad (n=1,2,\cdots)$$

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are easily estimated (in particular $|b_{k0}|\leqslant 2$). Indeed, twice application of integration by parts gives $b_{kn}=O(k^2/n^2)$ $(n\to+\infty)$ where the absolute constant implied by $O(\cdot)$ is independent of k and n.

Let us assume that $\sum_1^\infty f(k)k^2$ converges absolutely. Consider the absolutely convergent trigonometric series (with $y=\phi(x)$)

$$F(y)\equiv\sum_{k=1}^{\infty}f(k)\cos(ky)=\sum_{k=1}^{\infty}f(k)\cos(k\phi(x)), \tag{20}$$

where $F(y)$ is itself a Fourier series in $0<y<\pi$ with $\int_0^\pi F(y)\,dy=0$.

Now substituting (19) into the right-side of (20), we get

$$F(\phi(x))=\sum_{k=1}^{\infty}f(k)\left\{\frac{1}{2}b_{k0}+\sum_{n=1}^{\infty}b_{kn}\cos(nx)\right\}$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}f(k)b_{kn}\cos(nx)+B_0,$$

where the constant B_0 is given by

$$B_0=\frac{1}{2}\sum_{k=1}^{\infty}f(k)b_{k0}.$$

Note that the order of summation of the double series derived above can be reversed inasmuch as there is an absolute constant M such that

$$\left|\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}f(k)b_{kn}\cos(nx)\right|\leqslant\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|f(k)||b_{kn}|$$

$$\leqslant M\left(\sum_{k=1}^{\infty}|f(k)|\cdot k^2\right)\left(\sum_{n=1}^{\infty}1/n^2\right)<+\infty.$$

Thus we may express $F(y)\equiv F(\phi(x))$ as a convergent cosine series in x,

$$F(\phi(x))=\sum_{n=1}^{\infty}g(n)\cos(nx)+B_0;\quad(0<x<\pi)$$

with $g(n)$ being given by

$$g(n)=\sum_{k=1}^{\infty}f(k)b_{kn}=\sum_{k=1}^{\infty}f(k)\Phi_c(n,k). \tag{15 bis}$$

Now suppose that $\sum_1^\infty g(k)k^2$ also converges absolutely. Then starting with the cosine series for $F[\phi(x)]\equiv F\{\phi[\phi(y)]\}\equiv F(y)$ we find in an entirely like manner

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$$F(y)=\sum_{k=1}^{\infty}g(k)\cos[k\phi(y)]+B_0$$
$$=\sum_{n=1}^{\infty}f(n)\cos(ny)+B_1+B_0;$$

where $f(n)$ is given by

$$f(n)=\sum_{k=1}^{\infty}g(k)b_{k_n}=\sum_{k=1}^{\infty}g(k)\Phi_c(n,k),$$

and the constant B_1 is easily found to be

$$B_1=\frac{1}{\pi}\int_0^{\pi}[F(y)-B_0]dy=\frac{1}{\pi}\int_0^{\pi}F(y)dy-B_0=-B_0,$$

so that $B_1+B_0=0$, conforming with the original definition of $F(y)$ as given by (20). Hence the theorem is proved by (15 bis) and (16 bis).

Recall that $\phi(0)=\pi$, $\phi(\pi)=0$, so that $\sin[k\phi(0)]=\sin[k\phi(\pi)]=0$. Thus if we expand $\sin[k\phi(x)]$ into a convergent sine series

$$\sin[k\phi(x)]=\sum_{n=1}^{\infty}a_{k_n}\sin(nx),\quad (0<x<\pi)$$

with the Fourier coefficients a_{k_n} being given by

$$a_{k_n}=\frac{2}{\pi}\int_0^{\pi}\sin[k\phi(x)]\sin(nx)dx,$$

we can again obtain an estimate for $|a_{k_n}|$ by integration by parts; indeed, we have

$$\frac{\pi}{2}a_{k_n}=\frac{1}{n}\int_0^{\pi}\cos(nx)\frac{d}{dx}\sin[k\phi(x)]dx$$
$$=-\frac{1}{n^2}\int_0^{\pi}\sin(nx)\left(\frac{d}{dx}\right)^2\sin[k\phi(x)]dx=O(k^2/n^2),\quad (n\to\infty).$$

This ensures that the process for justifying (15)$\Longleftrightarrow$(16) may also be used(with minor modifications) to yield a proof of the following.

Theorem 7. With the same hypotheses as that of Theorem 6 we have the reciprocal formulas(17) and (18), provided that $\sum_1^{\infty}f(k)k^2$ and $\sum_1^{\infty}g(k)k^2$ are absolutely convergent series.

Finally, let us close this section with an open problem as follows: Given a self-reciprocal function $\phi(x)$ on $[0,\pi]$, is it possible to establish the reciprocal relations (15)$\Longleftrightarrow$(16) and (17)$\Longleftrightarrow$(18) under some weaker conditions than those of

$$\phi''(x)\in C_{[0,\pi]};\quad \sum_1^{\infty}|f(k)|k^2<+\infty,\quad \sum_1^{\infty}|g(k)|k^2<+\infty?$$

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§6 A formal Process for Obtaining Self-reciprocal Series Transforms

The basic idea employed in deriving self-reciprocal relations (15)$\Longleftrightarrow$(16) and (17)$\Longleftrightarrow$(18) can be generalized. Let $y=\phi(x)$ be a real self-reciprocal function defined on $[0, \lambda]$. Suppose that $\{\omega_k(x)\}$ is a complete orthonormal system of functions defined on $[0, \lambda]$ and that each $\omega_k(\phi(x))$ can be expanded into an absolutely convergent series (generalized Fourier series) of the form

$$\omega_k(\phi(x))=\sum_{n=0}^{\infty}c_{kn}\omega_n(x), \tag{21}$$

where the Fourier coefficients c_{kn} are determined by the inner products

$$c_{kn}=(\omega_k(\phi(x)),\ \omega_n(x)). \tag{22}$$

If for a given sequence $\{f(k)\}$ satisfying certain conditions we may derive formally

$$\sum_{k=0}^{\infty}f(k)\omega_k(y)=\sum_{k=0}^{\infty}f(k)\omega_k(\phi(x))$$
$$=\sum_{k=0}^{\infty}f(k)\Big(\sum_{n=0}^{\infty}c_{kn}\cdot\omega_n(x)\Big)=\sum_{n=0}^{\infty}g(n)\cdot\omega_n(x),$$

where the order of double summation is assumed reversible so that $g(n)$ is given by

$$g(n)=\sum_{k=0}^{\infty}c_{kn}\cdot f(k), \tag{23}$$

then we may obtain in a like manner

$$\sum_{n=0}^{\infty}g(n)\omega_n(x)=\sum_{k=0}^{\infty}g(k)\omega_k(\phi(y))=\sum_{n=0}^{\infty}f(n)\omega_n(y)$$

with $f(n)$ being given by

$$f(n)=\sum_{k=0}^{\infty}c_{kn}\cdot g(k). \tag{24}$$

This shows the self-reciprocal pair (23)$\Longleftrightarrow$(24) may be derived from (21) and (22) under suitable conditions.

Apparently the most important step of derivation to be justified is that of reversing the order of summations in the double series

$$\sum_k\ \sum_n f(k)c_{kn}\omega_n(x) \text{ and } \sum_k\ \sum_n g(k)c_{kn}\omega_n(x).$$

Generally this may be verified for special systems $\{\omega_k(x)\}$ and suitably restricted sequences $\{f(k)\}$ and $\{g(k)\}$ with an appropriate estimate for $|c_{kn}|$.

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§7 A Kind of Self-reciprocal Integral Transforms

The essential idea employed in establishing the self-reciprocal formulas (1) and (2) can be equally effectively applied to constructing a kind of self-reciprocal integral transforms. What we shall give below is merely one of possible analytic processes that can lead to self-reciprocal integral transforms or the like.

In what follows we always assume that $y=\phi(x)$ is a real-valued function self-reciprocal on $0\leqslant x\leqslant 1$ and infinitely differentiable in $0<x<1$. Let $f(x)$ and $g(x)$ be two real functions Lebesgue integrable on $(0,\infty)$, i. e. $f\in L(0,\infty)$ and $g\in L(0,\infty)$. Suppose that $f(x)$ and $g(x)$ are related by the following equation

$$F(t)\equiv\int_0^\infty f(s)e^{-ts}ds=\int_0^\infty g(s)(\phi(e^{-t}))^s ds \tag{25}$$

where $t\geqslant 0$. Note that $\phi(0)=1$, $\phi(1)=0$ and $|\phi(x)|\leqslant 1$, $(0\leqslant x\leqslant 1)$, so that we may denote $e^{-u}=\phi(e^{-t})$ $(t\geqslant 0)$. The self-reciprocity of ϕ gives

$$e^{-t}=\phi(\phi(e^{-t}))=\phi(e^{-u}).$$

Thus (25) is equivalent to the equation

$$G(u)\equiv\int_0^\infty f(s)(\phi(e^{-u}))^s ds=\int_0^\infty g(s)e^{-us}ds. \tag{26}$$

Clearly the Laplace integrals appearing in (25) and (26) are both uniformly convergent and absolutely convergent for all $t\geqslant 0$ and $u\geqslant 0$, respectively, since $f\in L$ and $g\in L$.

Let us now apply the well-known Post-Widder inversion operator to the left-side of (25), obtaining

$$f(u)=\lim_{m\to\infty}\frac{(-1)^m}{m!}F^{(m)}\left(\frac{m}{u}\right)\left(\frac{m}{u}\right)^{m+1},(0<u<\infty),$$

where the inversion operation may be written as $f=\mathscr{L}^{-1}(F)$.

In accordance with Widder we may denote for simplicity

$$L_{m;u}[F(t)]\equiv\frac{(-1)^m}{m!}\left(\frac{m}{u}\right)^{m+1}D_t^m F(t)\,|_{t=(m/u)},$$

so that the inversion operator $\mathscr{L}^{-1}$ is defined by

$$\mathscr{L}^{-1}(F)=\lim_{m\to\infty}L_{m;u}[F(t)].$$

Consequently we may infer from (25) and (26) the following inverse relations

$$f(u)=\mathscr{L}^{-1}(F)=\mathscr{L}^{-1}\left\{\int_0^\infty g(s)(\phi(e^{-t}))^s ds\right\}, \tag{27}$$

$$g(u)=\mathscr{L}^{-1}(G)=\mathscr{L}^{-1}\left\{\int_0^\infty f(s)(\phi(e^{-t}))^s ds\right\}. \tag{28}$$

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Since the Laplace integrals expressed by (25) and (26) can be differentiated under integral sign (with respect to $t>0$ and $u>0$ respectively), we see that the operator $\mathscr{L}^{-1}$ may be placed under integral sign. Thus from (28) and (27) we may derive a pair of self-reciprocal relations as follows

$$g(u)=\lim_{m\to\infty}\int_0^\infty H(s,u;m)f(s)ds, \tag{29}$$

$$f(u)=\lim_{m\to\infty}\int_0^\infty H(s,u;m)g(s)ds, \tag{30}$$

where the kernel $H(s,u;m)$ of integral transformation is defined by

$$H(s,u;m)=\frac{(-1)^m}{m!}\left(\frac{m}{u}\right)^{m+1}\left(\frac{d}{dt}\right)^m(\phi(e^{-t}))^s\Big|_{t=(m/u)},$$

or, in abbreviation,

$$H(s,u;m)=L_{m,u}[(\phi(e^{-t}))^s]. \tag{31}$$

However, all what given above is merely a formal deduction. We shall now investigate some conditions under which the validity of the above deduction can be guaranteed. Given ϕ and f with $f\in L(0,\infty)$, we have to show that the integral equation (25) has a unique solution $g(s)$. Notice that (26) is equivalent to(25), so that asserting the existence of $g(s)\in L(0,\infty)$ is equivalent to affirming that the integral

$$G(u)\equiv\int_0^\infty f(s)(\phi(e^{-u}))^s ds$$

can be represented as a Laplace-Lebesgue integral involving $g(s)$ as its determining function. Evidently a well-known representation theorem of Widder's (cf. [8] chap. 7, §17) applies to the present case.

According to Widder, a function $\psi(t)$ is said to satisfy "Condition D", if the following 3 conditions are fulfilled:

(1) $\psi(t)$ is infinitely differentiable in $(0,\infty)$ with $\psi(\infty)=0$;

(2) for every positive integer m, the function $L_{m,u}[\psi(t)]$ is Lebesgue integrable, i. e.

$$\int_0^\infty |L_{m,u}[\psi(t)]|du<+\infty;$$

(3) the sequence $\{L_{m,u}[\psi(t)]\}$ converges in mean with index one, i. e.

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\int_0^\infty |L_{m,u}[\psi]-L_{n,u}[\psi]|du=0.$$

Now suppose that $G(v)\equiv\int_0^\infty f(s)(\phi(e^{-v}))^s ds$ satisfies Condition D. Then Widder's representation theorem ([8], Theorem 17, p. 318) affirms that there is a function $g(s)\in L(0,\infty)$ such that

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$$G(v)=\int_0^\infty g(s)e^{-vs}ds. \tag{32}$$

This ensures that both (26) and (25) are valid, and consequently we have (27) and (28). More precisely, Widder's inversion theorem imp ies that the reciprocal relations (27) and (28) do hold for all values $u>0$ belonging to the Lebesgue sets for f and g, respectively. Summing up, we have the following.

Theorem 8. Let $\phi(x)$ be a self-reciprocal function defined on $[0, 1]$ and having derivatives of all orders in $(0, 1)$. Let $H(s, u; m)$ be defined by (31). Then for every $f(s)\in L(0, \infty)$ such that the function $G(t)\equiv\int_0^\infty f(s)(\phi(e^{-t}))^s ds$ satisfies Condition D, we have self-reciprocal relations (29) and (30) which are valid for all positive u in the Lebesgue sets for g and f respectively.

Example. Take two simple self-reciprocal functions

$$y=\phi_1(x)=1-x; \qquad y=\phi_2(x)=(1-x^2)^{1/2}, \quad (0\leqslant x\leqslant 1).$$

Correspondingly we get two kernels as follows

$$H_1(s,u;m)=\frac{(-1)^m}{m!}\left(\frac{m}{u}\right)^{m+1}D_t^m(1-e^{-t})^s\big|_{t=(m/u)};$$

$$H_2(s,u;m)=\frac{(-1)^m}{m!}\left(\frac{m}{u}\right)^{m+1}D_t^m(1-e^{-2t})^{s/2}\big|_{t=(m/u)}.$$

As a final remark it may be worthy of mention that self-reciprocal relations can also be obtained by making use of Fourier transforms instead of Laplace transforms. For instance, suppose that $y=\phi(x)$ is a self-reciprocal function defined on $(-\infty, \infty)$ with $\phi(-\infty)=+\infty$, $\phi(+\infty)=-\infty$. We may consider a pair of equivalent equations as follows

$$F(x)\equiv\int_0^\infty f(s)\cos(sx)ds=\int_0^\infty g(s)\cos(s\phi(x))ds,$$

$$G(y)\equiv\int_0^\infty f(s)\cos(s\phi(y))\,ds=\int_0^\infty g(s)\cos(sy)ds.$$

Certainly we may apply Fourier inversion formulas to the above equations with suitable restrictions upon f or g, thus arriving at a pair of self-reciprocal relations

$$=f(u)\frac{2}{\pi}\int_0^\infty \cos(ut)dt\int_0^\infty g(s)\cos(s\phi(t))ds, \tag{33}$$

$$g(u)=\frac{2}{\pi}\int_0^\infty \cos(ut)dt\int_0^\infty f(s)\cos(s\phi(t))ds. \tag{34}$$

Obviously this kind of reciprocal relations may be regarded as an integral analogue of (15) and (16), and the reciprocity $(33)\Longleftrightarrow(34)$ can only be established for special classes of functions $\{f\}$ and $\{g\}$. Detailed discussion may be left to the interested reader.

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自反函数与自反变换

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中文摘要

这篇论文的主题是研究由自反函数导出自反级数变换与自反积分变换的一般方法。方法的基本思想来源于著者1964年发表的一篇文章(美国数学家H. W. Gould曾在德国的Zentralblatt杂志上介绍了该文的主要结果，见该刊1973年254卷#44003)。现在这个工作较系统地发展了早先提出的思想方法，建立了若干一般性定理，有些定理可以作为构造各种级数变换互反公式的方法原则。文中除定理1,2,8之外大部份结果是属于初次发表。值得指出的是，本文中探讨的方法与概念，都是根据对某些特例的分析形成的，积分自反变换方面的结果是以级数变换为借镜，通过类比法得到的。(本文基本内容曾在1981年4月20日大连工学院举办的科学报告会上报告过)。

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