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On a General Probabilistic Representation Formula for Semigroups of Operators*

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Abstract

In this paper we present a general probabilistic representation theorem for semigroups of operators which essentially comprises almost all known representation theorems of this kind. Especially, so-called first and second main theorems turn out to be special cases of the same general formula. The general theorem also permits the calculation of rates of convergence involving the second modulus of continuity.

1. Introduction

It is the purpose of this paper to present a general probabilistic representation theorem for semigroups of operators which extends a similar theorem due to Chung ([4], Theorem 5). We will show that almost all known representation formulas for operator semigroups are essentially obtained from this by specialisation; also,

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so-called first and second main theorems turn out to be special cases of the same general formula. Some of the ideas used in the proof are implicit in the work of Chung [4] and Butzer and Hahn [3], hence it will also be possible to calculate rates of convergence from the general theorem involving the second modulus of continuity.

In the sequel we will extensively make use of probability theory in Banach spaces. For a nice survey of the field we refer to the introducing chapters of Bharucha-Reid's book [1].

2. Notations and preliminaries

We consider a strongly continuous one-parameter semigroup of bounded linear operators $\mathscr{T}=\{T(t);\ t\geqslant 0\}$ with values in a Banach space $\mathscr{X}$ (see Butzer and Berens [2] for definitions) endowed with the Borel σ-field $\mathscr{B}$ (i. e. the σ-field generated by the norm-topology of $\mathscr{X}$). In this case, $\mathscr{T}$ is a commutative Banach algebra where the composition of mappings plays the role of the product, and every T(t) $\in \mathscr{T}$ is both strongly and weakly measurable and separably valued (cf. also Butzer and Hahn [3]). For a (possibly $\mathscr{T}$-valued) random variable (r. v.) X E(X) will denote the expectation (which possibly is a Bochner- or pettis-integral); P will denote the underlying probability measure.

3. The Representation Theorem

We begin with a version of the Fubini theorem for Banach algebras:

Lemma Let $\mathscr{S}$ denote an arbitrary (not necessarily commutative) Banach algebra with product $\circ$. Let further denote $\mathscr{B}$ $(\mathbb{R})$ the Borel σ-field over the real numbers $\mathbb{R}$, and let μ and ν be σ-finite Borel-measures on $\mathscr{B}$ $(\mathbb{R})$. Then for all separably valued Borel-measurable mappings f, g from $\mathbb{R}$ into $\mathscr{S}$ such that f is μ-integrable and g is ν-integrable the (measurable) mapping h defined by

$h(s,t)=f(s)\circ g(t)$, $s,t\in\mathbb{R}$ is $\mu\otimes\nu$-integrable (where $\mu\otimes\nu$ denotes the product measure), and

$$\int h d\mu\otimes\nu=\int f d\mu\circ\int g d\nu. \tag{1}$$

Proof. This follows immediately from a simple extension of Mikusiński's ([5]) Theorem 6.4 to the case of arbitrary σ-finite Borel-measures; alternatively, use Bharucha-Reid's ([1]) Theorem 1.11 or proceed by algebraic induction proving the Lemma for simple functions first.

Corollary 1. Let X and Y be independent real r. v. 's and f, g as in the Lemma such that $f(X)$ and $g(Y)$ are integrable. Then

$$E[f(X)\circ g(Y)] = E[f(X)]\circ E[g(Y)]. \tag{2}$$

Now we are able to formulate the general representation theorem:

Theorem. Let N be a non-negative integer-valued r. v. and X be a non-negative real r. v. such that the characteristic functions ϕ_N and ϕ_X of N and X resp. (i. e., the Fourier transforms of their distributions) are analytic in some neighbourhood of the origin. Let further ψ_N denote the moment-generating function of N. Then for sufficiently large n, $\psi_N\left(E\left[T\left(\frac{X}{n}\right)\right]\right)$ is a bounded linear operator, and with $\xi = E(N)E(X)$, we have

$$T(\xi) = \lim_{n\to\infty}\left\{\psi_N\left(E\left[T\left(\frac{X}{n}\right)\right]\right)\right\}^n \tag{3}$$

where the limit is to be understood in the strong operator topology.

Proof. we will first prove the Theorem for $N\equiv 1$; then we have to show

$$T(\xi) = \lim_{n\to\infty}\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^n \quad \text{with } \xi = E(X). \tag{4}$$

Let $\{X_n;\ n\in \mathbb{N}\}$ be independent copies of X. It is known that there exist constants $M, \omega>0$ such that

$$\|T(t)\|\leqslant Me^{\omega t},\quad t\geqslant 0 \text{ (see [2])}. \tag{5}$$

hence for every $r\geqslant 1$ and sufficiently large n,

$$E\left[\left\|T\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)\right\|^r\right]\leqslant ME\left[\exp\left(\frac{\omega r}{n}\sum_{k=1}^{n}X_k\right)\right] = M\left\{E\left[\exp\left(\frac{\omega r}{n}X\right)\right]\right\}^n$$

$$= M\phi_X^n\left(-\frac{i\omega r}{n}\right) = M\left(1+\frac{\omega r\xi}{n}+O\left(\frac{1}{n^2}\right)\right)^n \to Me^{\omega r\xi} \quad \text{for } n\to\infty$$

which also indicates that $E\left[T\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)\right]$ is bounded for large n; likewise for $E\left[T\left(\frac{X}{n}\right)\right]$.

Since $\frac{1}{n}\sum_{k=1}^{n}X_k\to\xi$ almost surely by the law of large numbers (LLN), Lemma 1 of Chung ([4]) is applicable giving

$$T(\xi) = \lim_{n\to\infty}E\left[T\left(\frac{1}{n}\sum_{k=1}^{n}X_k\right)\right] = \lim_{n\to\infty}E\left[\prod_{k=1}^{n}T\left(\frac{X_k}{n}\right)\right]$$

$$= \lim_{n\to\infty}\prod_{k=1}^{n}E\left[T\left(\frac{X_k}{n}\right)\right] = \lim_{n\to\infty}\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^n \tag{6}$$

by (2) and the semigroup-property of the commutative Banach algebra $\mathcal{T}$.

Now for the general case, let $Y=\sum_{k=1}^{N}X_k$ where $\{X_n; n\in \mathbb{N}\}$ again are independent copies of X, independent of N. Since $\psi_N=\phi_N$ (-i log (.)) and $\phi_Y=\psi_N(\phi_X)$, ϕ_Y is analytic in some neighbourhood of the origin. Another application of the Fubini theorem for Banach algebras gives

$$E\left[T\left(\frac{Y}{n}\right)\right]=E\left[\prod_{k=1}^{N}T\left(\frac{X_k}{n}\right)\right]=\sum_{m=0}^{\infty}E\left[\prod_{k=1}^{m}T\left(\frac{X_k}{n}\right)\right]P(N=m)$$

$$=\sum_{m=0}^{\infty}\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^mP(N=m)=\psi_N\left(E\left[T\left(\frac{X}{n}\right)\right]\right) \tag{7}$$

for sufficiently large n, where $\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^0=I$ (the identity operator); hence (3) follows from (6), replacing X by Y (note that $E(Y)=E(N)E(X)$).

Corollary 2 Let N be as in the Theorem with $E(N)=\xi$. Then

$$T(\xi)=\lim_{n\to\infty}\left\{\psi_N\left(T\left(\frac{1}{n}\right)\right)\right\}^n=\lim_{n\to\infty}\{\psi_N(nR(n))\}^n \tag{8}$$

where $R(\lambda)=\int_0^\infty e^{-\lambda t}T(t)dt$, $\lambda>0$ denotes the resolvent of the semigroup (see [2]).

Proof. Let simply $X\equiv 1$ in (3); for the other part, let X have an exponential distribution with unit mean, then clearly $E\left[T\left(\frac{X}{n}\right)\right]=n\ R(n)$.

As can be seen by Corollary 2, first main theorems (i. e. theorems involving $T\left(\frac{1}{n}\right)$) and second main theorems (i. e. theorems involving $R(n)$) are both obtained from the general theorem by specialisation of the distribution of X.

By further specialisation of the distributions of N and X we reobtain the following well-known representation formulas:

A) If N has a Poisson distribution with mean ξ, then $\psi_N(t)=e^{-\xi}e^{t\xi}$, hence

$$T(\xi)=\lim_{n\to\infty}\left\{\psi_N\left(T\left(\frac{1}{n}\right)\right)\right\}^n=\lim_{n\to\infty}\ \exp\left(\xi n\left[T\left(\frac{1}{n}\right)-I\right]\right)$$

(Hille)

$$T(\xi)=\lim_{n\to\infty}\{\psi_N(nR(n))\}^n=\lim_{n\to\infty}\ \exp(-\xi nI+\xi n^2R(n))$$

(Phillips)

B) If N has a binomial distribution over $\{0,1\}$ with mean ξ, then $\psi_N(t)=1$

$-\xi+\xi t$, hence

$$T(\xi)=\lim_{n\to\infty}\left\{\psi_N\left(T\left(\frac{1}{n}\right)\right)\right\}^n=\lim_{n\to\infty}\left(I-\xi+\xi T\left(\frac{1}{n}\right)\right)^n$$

(Kendall)

$$T(\xi)=\lim_{n\to\infty}\{\psi_N(nR(n))\}^n=\lim_{n\to\infty}(I-\xi+\xi nR(n))^n$$

(Chung)

C) If N has a geometric distribution with mean ξ, then $\psi_N(t)=(1+\xi-\xi t)^{-1}$, hence

$$T(\xi)=\lim_{n\to\infty}\left\{\psi_N\left(T\left(\frac{1}{n}\right)\right)\right\}^n=\lim_{n\to\infty}\left(I+\xi-\xi T\left(\frac{1}{n}\right)\right)^{-n}$$

(Shaw [6], (30))

$$T(\xi)=\lim_{n\to\infty}\{\psi_N(nR(n))\}^n=\lim_{n\to\infty}(I+\xi-\xi nR(n))^{-n}$$

(Chung)

D) If $N\equiv 1$, $X=\xi Y$ where Y is geometrically distributed with mean 1, then

$$T(\xi)=\lim_{n\to\infty}\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^n=\lim_{n\to\infty}\left(2I-T\left(\frac{\xi}{n}\right)\right)^{-n}$$

(Shaw [6], (40))

E) If $N\equiv 1$ and X has an exponential distribution with mean ξ, then

$$T(\xi)=\lim_{n\to\infty}\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^n=\lim_{n\to\infty}\left(\frac{n}{\xi}R\left(\frac{n}{\xi}\right)\right)^n$$

(Post-widder)

F) If $N\equiv 1$ and X has a gamma-distribution with density

$$f_X(t)=\frac{t^{\xi-1}}{\Gamma(\xi)}e^{-t},\quad t>0, \text{ then}$$

$$T(\xi)=\lim_{n\to\infty}\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^n=\lim_{n\to\infty}\left\{\frac{n^\xi}{\Gamma(\xi)}\int_0^\infty t^{\xi-1}e^{-nt}T(t)dt\right\}^n.$$

This is another form of Theorem 4 b) in Butzer and Hahn [3].

There is of course a very much larger variety of representation theorems which can immediately be deduced from the general theorem; also, uniform convergence statements and rates of convergence can easily be derived in the general case since the validity of (3) is essentially due to the LLN which is also basic for the work of Butzer and Hahn [3]. For example, if $N\equiv 1$ and X is uniformly distributed over $(0, 2\xi)$,

H) $$T(\xi)=\lim_{n\to\infty}\left\{E\left[T\left(\frac{X}{n}\right)\right]\right\}^n=\lim_{n\to\infty}\left\{\frac{n}{2\xi}\int_0^{2\xi/n}T(t)dt\right\}^n$$

with

$$\left\|\left\{\frac{n}{2\xi}\int_0^{2\xi/n}T(t)dt\right\}^n f-T(\xi)f\right\|<K\omega_2\left(\sqrt{\frac{7}{3n}}\xi,\ f\right),\ f\in\mathscr{X}$$

where K is a positive constant and ω_2 denotes the second modulus of continuity (cf. Butzer and Hahn [3], Theorem 1).

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