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A Generalization of Univariate Splines with Equally Spaced Knots to Multivariate Splines*

Charles K. Chui** Ren-Hong Wang+

Department of Mathematics
Texas A&M University
College Station
Texas 77843 USA

Department of Mathematics
Jilin University
Changchun
Jilin, China

Since the unit interval $I=[0,1]$ can also be considered as a 1-simplex in $\mathbf{R}^1$, it is natural to consider two different generalizations of I in the higher dimensional setting, namely: the unit cubes or simplices. For example, in $\mathbf{R}^2$ we have the unit square I^2 or the two 2-simplices $\delta_1: 0\leqslant x\leqslant y\leqslant 1$ and $\delta_2: 0\leqslant x\leqslant y\leqslant 1$. To partition $\mathbf{R}^1$ into equally spaced intervals, I is shifted repeatedly by one unit at a time both to the left and to the right. Similarly, if I^2 is used to partition $\mathbf{R}^2$, we have the rectangular grid partition:

$$\Pi:\ x=i,\ y=i,\ i=\ldots,-1,0,1,\ldots;$$

and if the 2-simplices δ_1 and δ_2 are used, we have the triangular grid partition:

$$\Delta:\ x=i,\ y=i,\ x-y=i,\ i=\ldots,-1,0,1,\ldots.$$

On the other hand, there are two different generalizations of polynomials of one variable to polynomials of several variables, namely: polynomials of certain coordinate degree, or polynomials of certain total degree. For example, a polynomial in two variables of coordinate degree (m,n) is

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$$p_{m,n}(x,y)=\sum_{j=0}^{n}\sum_{i=0}^{m}a_{ij}x^iy^j;$$

while a polynomial in two variables of total degree k is

$$p_k(x,y)=\sum_{i+j\leq k}b_{ij}x^iy^j,$$

where the sum is taken over all non-negative integers i and j with $i+j\leqslant k$. The spaces of such polynomials will be denoted by $\mathbf{P}_{m,n}$ and $\mathbf{P}_k$, respectively.

Similarly, there are two different genralizatrons of univariate splines to spline functions of several variables, namely multivariate splines of certain coordinate degree, or multivariate splines of certain total degree. Therefore, if the definition of univariate splines with knots at the integers is generalized to the bivariate setting, it would seem natural to use the rectangular grid partition Π for bivariate splines of coordinate degree (m,n), and the triangular grid partition Δ for bivariate spline functions of total degree k. Similar grid partitions can be given to the multivariate setting. In this paper, we will only concentrate on the discussion of bivariate splines. Since the theory of bivariate splines with coordinate degree (m,n) on the rectangular grid partition Π is well understood (see, for example, de Boor [1]), we will only study bivariate splines in $C^\mu(\mathbf{R}^2)$ of total degree k on the triangular grid partition Δ. (See Fig. 1) The space of all such functions will be denoted by $S_k^\mu=S_k^\mu(\Delta)$.

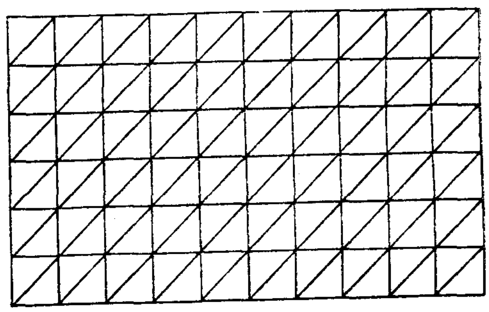

Fig. 1

It is well-known (cf. [8]) that univariate splines with knots at the integers are, on each bounded interval, linear combinations of translates of a single spline function known as a fundamental spline (or more commonly called a B-spline). Similarly, every bivariate spline function of certain coordinate degree with grid partition Π is again, on each bounded set in $\mathbf{R}^2$, a linear combination of translates of a single tensor-product B-spline function (cf. [1]). However, the analogous problem in the study of bivariate spline functions of total degree k, say $S_k^\mu(\Delta)$, is much more complicated and interesting. In the first place, there does not necessarily exist a nontrivial locally supported function in $S_k^\mu(\Delta)$. In fact, a necessary condition for its existence is the condition

$$k>\frac{3\mu+1}{2}\tag{1}$$

(cf. [4]). That is, the degree must be large enough when we consider splines in $C^\mu(\mathbf{R}^2)$. Since splines with the lowest degree are more desirable, we will consider the spaces

$$S^{\mu}:=S^{\mu}_{k(\mu)}$$

werhe $k(\mu)$ is the smallest integer k that satisfies (1). Hence, $S^0=S^0_1$, $S^1=S^1_3$, $S^2=S^2_4$, $S^3=S^3_6,\ldots$.

Let us be more precise on the notion of a local support of a bivariate spline function. Every curve we consider in this paper will be a polygonal Jordan curve consisting of segments of the grid partition Δ. The collection of all such curves will be denoted by Γ. A curve γ in Γ will be called a local supporting curve, and the region it encloses will be called a local support, of an $s\in S^{\mu}$, if s vanishes everywhere outside γ. We will use the notation $\gamma_1\prec\gamma_2$, where $\gamma_1,\gamma_2\in\Gamma$, if $\gamma_1\neq\gamma_2$ and the region enclosed by γ_1 is also enclosed by γ_2. Also $\gamma\in\Gamma$ will be called a minimal local supporting curve for S^{μ}, and the region it encloses will be called a minimal local support for S^{μ}, if it is a local supporting curve of some nontrivial $s\in S^{\mu}$, and for every $\gamma_1\prec\gamma$, γ_1 is not a local supporting curve of any nontrivial s in S^{μ}. If γ is a minimal local suppoiting curve for S^{μ} and is a local supporting curve of a nontrivial $B\in S^{\mu}$, say, we will call B a fundamental spline of S^{μ}.

Two curves γ_1 and γ_2 are said to be congruent to each other if γ_1 can be obtained from γ_2 by some translation $x\to x-a$ and $y\to y-b$. Hence, all curves in Γ that are congruent to some γ form an equivalence class, and will be considered as the same curve γ. Similarly, if two fundamental splines have the same (minimal) supporting curve and one is a constant multiple of the other, we will call them equivalent; and if all fundamental splines with the same supporting curve γ are equivalent, we will say that γ supports only "one" fundamental spline, B, say. Let $n(\mu)$ denote the number of (equivalence classes of) minimal supporting curves for S^{μ}. We have the following result.

THEOREM 1. $n(0)=1$, $n(1)=2$, and $n(2)=1$; and the minimal supporting curves for S^0, S^1, S^2, denoted by γ_0, γ^1_1, γ^2_1, γ_2 are given in Fig. 2, Fig, 3a, b, Fig. 4, respectively. Furthermore, each of these four curves supports only "one" fundamental spline $B^0\in S^0$, $B^{1,1}\in S^1$, $B^{1,2}\in S^1$, $B^2\in S^2$ respectively; and all of these fundamental splines are of one sign (either strictly positive or strictly negative) inside the supporting curves.

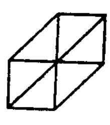

Fig. 2

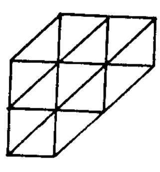

Fig. 3a

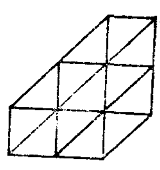

Fig. 3b

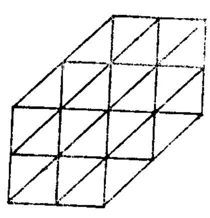

Fig. 4

We will normalize the above fundamental splines to be nonnegaive. The result for $\mu=0$ is trivial but is included for comparison. To express the fundamental splines it is more convenient to use barycentric coordinates; that is, on a triangle with vertices A, B and C, we define a linear polynomial $a\in P_1$ by requiring $a(A)=1$ and $a(B)=a(C)=0$, and the linear polynomials b and c are defined similarly. Note that $a+b+c\equiv 1$, so that any two of the polynomials a, b, c can be used as independent variables in place of x and y. Representations of $B^{1,1}$ and B^2 are given in [6] (where there are minor misprints in each case). $B^{1,2}$ is the same as $B^{1,1}$ in barycentric coordinates. We list B^0, $B^{1,1}$ and B^2 below, using letters P, Q, R, respectively, to denote their polynomial pieces, and subscripts to indicate the triangles on which the polynomials are defined. (See Fig. 5a, b, c).

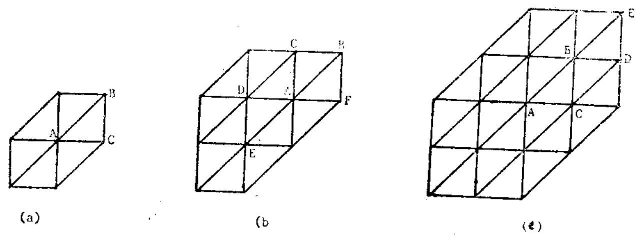

Fig. 5

(2) $$P_{ABC}=a$$

(3)
$$\begin{cases} \text{(corner)} & Q_{ABF}=a^3 \\ \text{(edge)} & Q_{ABC}=a^2(a+3c) \\ \text{(interior)} & Q_{ADC}=3(a+d)^2-2(a^3+d^3)-3ad(a+d) \\ \text{(center)} & Q_{ADE}=1+3(a+d)-3(a^2+d^2)-3ad(a+d) \end{cases}$$

(4)
$$\begin{cases} \text{(edge)} & R_{BDE}=b^3(b+2d) \\ \text{(interior)} & R_{BCD}=2(b+c)^3-(b^4+c^4)-2(b^3c+bc^3) \\ \text{(cenrer)} & R_{ABC}=6-12(b^2+bc+c^2)+8(b^3+c^3) \\ & \qquad +12(b^2c+bc^2)-(b^4+c^4)-2(b^3c+bc^3). \end{cases}$$

The other polynomial pieces are identical in barycentric coordinates, depending on the type of triangles (corner, edge, interior, center) on which they are defined.

We next consider the linear independency and the spans of the translates of the above fundamental splines. To be more precise, we will assume that each of the (minimal) supporting curves γ_0, γ_1^1, γ_1^2, and γ_2 encloses the origin, say, at point A in Fig. 5a,c and a point D in Fig. 5b; and use the notaton $B_{ij}=B_{ij}(x,y)=B(x-i, y-j)$ for any given function $B=B(x,y)$. Let m and n be positive integers, and consider the rectangular region

$$D=D_{mn}=\{(x,y):\ 0\leqslant x\leqslant m,\ 0\leqslant y\leqslant n\}.$$

We need the index sets:

$$\Omega_p = \{(i,j): B^p_{i\,j} \text{ does not vanish identically on } D\}$$

$$\Omega^q_1 = \{(i,j): B^{1,q}_{i\,j} \text{ does not vanish identically on } D\},$$

where $p = 0,2$ and $q = 1,2$. As usual, let $S^\mu(D) = S^\mu_{K(\eta)}(\Delta, D)$ be the space of all bivariate spline functions $s \in S^\mu$ restricted on D. We have the following result.

THEOREM 2 (*a*) $A_0 = \{B^0_{i\,j}: (i,j) \in \Omega_0\}$ is a basis of $S^0(D)$.

(*b*) For each $q = 1, 2$, $A^q_1 = \{B^{1,q}_{i\,j}: (i, j) \in \Omega^q_1\}$ is a linearly independent set on D but does not span $S^1(D)$.

(*c*) $A_1 = A^1_1 \cup A^2_1$ is a linearly dependent set on D and spans $S^1(D)$. Furthermore, the set

$$A_1 \setminus \{B^{1,1}_{m,n+1}, B^{1,2}_{m+1,n}, B^{1,2}_{m+1,n-1}\}$$

is a basis of $S^1(D)$.

(*d*) $A_2 = \{B^2_{i\,j}: (i, j) \in \Omega_2\}$ is a linearly independeot set on D but does not span $S^2(D)$.

In view of (*d*), it is natural to ask if there are other locally supported spline functions in S^2 that, together with A_2, would span $S^2(D)$. The answer is negative, and in fact, we have the following.

THEOREM 3 Every locally supported bivariate spline function in S^2 is a linear combination of some translates of the fundamental spline B^2.

Hence, to give a basis of $S^2(D)$, we need some non-locally supported functions. We have the following.

THEOREM 4 A basis of $S^2(D)$ is given by

$$C = \{B^2_{i\,j}, (x-p)^4_+, (y-q)^4_+, (x-y-r)^4_+:$$
$$(i, j) \in \Omega_2,\ p = 0, \dots, m-1,\ q = 0, \dots, n-1, \text{ and } r = -n, \dots, m-1\},$$

In view of Theorem 3, we consider the subspace lS^μ of all locally supported functions in S^μ and let $lS^\mu(D)$ be the subspace of $S^\mu(D)$, consisting of all bivariate spline functions in $S^\mu(D)$ which are restrictions on D of functions in lS^μ. From the above results, we see that $\dim lS^2(D) = (m+3)(n+3) - 2$ while it is known (cf. [9], [4]) that $\dim S^2(D) = (m+5)(n+5) - 18$. (The dimensions of bivariate spline spaces on fairly general grid partitions can be found in [4].)

We conclude this paper by posing some problems.

CONJECTURE (*a*) $n(\mu) = 1$ if μ is even and $n(\mu) = 2$ if μ is odd, (*b*) All fundamental splines are of one sign inside the minimal supporting curves, and each minimal supporting curve supports only "one" fundamental spline. (*c*) The collection of all nontrivial translates in $S^\mu(D)$ of a single fundamental spline is a linearly independent set on D. (*d*) Every locally supported bivariate spline function in S^μ is a linear combination of some translates of the fundamental splines in S^μ.

The collection of all nontrivial translates on a rectangle, say D, of a single bivariate "B-spline" are in general linearly dependent (cf, [2], [3], [5], [7]). However, the spaces $S^\mu(D)$ we consider in this paper are quite special, since we believe that they are natural generalizations of the spaces of univariate splines with equally spaced knots. Recently, translates of multivariate "B-splines" have been used for approximation of functions of several variables. In view of Theorems 2(d) and 3, however, it is interesting to study how much better do bivariate splines approximate. For $\mu=2$, one could use the basis C of $S^2(D)$ to compare with the approximation properties of the basis A_2 of $lS^2(D)$. Many similar problems concerning the spaces S^μ and $S^\mu(D)$ can be posed. Proofs of the theorems in this paper and other related results will appear elsewhere.

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作为一元等距结点样条推广的多元样条

崔锦泰　　　　王仁宏

（美国德克萨斯A&M大学）（吉林大学）

摘　要

作者们在[4]中已经指出了给定剖分下多元 B-样条存在的必要条件（1）。它表明，并不是对所有的剖分都有多元 B-样条存在的。人们也许以为，如同一元情况一样，只要多元B-样条存在，则它们一定组成多元样条空间的支集（即多元样条空间是所有多元B-样条所支架起来的空间）。本文以标准的三角剖分（2-单纯形）下，多元样条空间 $S^2:=S^2_4$ 为例指出这种认识是错误的。事实上，本文定理 2，3 和 4 对这个问题已给出了明确的结论。

以上结论说明多元 B-样条并不是基本的。于是本文引进了多元样条空间的新子空间 lS^μ，并采用 Schoenberg 开创性文章[8]中所用的"基本样条"（Fundemental spline）一词，而引进多元基本样条的新概念。定理 1 还给出了基本样条的某些性质。

本文最后还给出了一些有关猜想。