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关于von Mises统计量Berry-Esseen定理的一点注记*

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设$X_1,X_2,\cdots,X_n$为独立、同分布随机变量，$\varphi(x_1,x_2,\cdots,x_m)$为$m$元对称函数. 那末，U-统计量$U_n$和von Miess统计量$V_n$分别定义作：

$$U_n=\binom{n}{m}^{-1}\sum_{1\leqslant\alpha_1<\alpha_2\cdots<\alpha_m\leqslant 1}\varphi(X_{\alpha_1},X_{\alpha_2},\cdots,X_{\alpha_m})$$

$$V_n=n^{-m}\sum_{\alpha_1,\alpha_2,\cdots,\alpha_m=1}^{n}\varphi(X_{\alpha_1},X_{\alpha_2},\cdots,X_{\alpha_m})$$

文[1]曾利用U_n与V_n的关系，得到V_n的Berry-Esseen定理，即

定理1 设V_n为以φ为核的von Mises统计量，假如φ满足：

1. 对任何$\alpha_j=1,2,\cdots,m$, $j=1,2,\cdots,m$成立

$$E|\varphi(X_{\alpha_1},X_{\alpha_2},\cdots,X_{\alpha_m})|<\infty \tag{1}$$

2.
$$E|\varphi(X_1,X_2,\cdots,X_m)|^3<\infty \tag{2}$$

3.
$$E|\varphi(X_1,X_1,X_3,\cdots X_m)|^{\frac{3}{2}}<\infty \tag{3}$$

则若以$F_n(x)$记$\sqrt{n}(V_n-\theta)/m\sigma$的分布函数，$\Phi(x)$为标准正态分布函数，则

$$\|F_n-\Phi\|=\sup_x|F_n(x)-\Phi(x)|=O(n^{-\frac{1}{2}}) \tag{4}$$

此处

$$\theta=E\varphi(X_1,X_2,\cdots,X_m), \tag{5}$$

$$\sigma^2=E[\varphi(X_1,X_2,\cdots,X_m)-\theta]^2. \tag{6}$$

文[1]在得出这一定理的同时，就曾预言，为使定理1成立，条件(3)可能不是必

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要的，作者在学习[1]的基础上发现，定理1的条件（3）确实是不必要的，这一想法得到陈希孺教授的肯定和指正。以下欲证

定理2 设V_n为以φ为核的von Mises统计量，假如φ满足条件（1）、(2)，则(4)式成立。

为证此定理，先引进文[1]的结论：

假如φ满足条件（1），则

$$V_n = U_n + \frac{a}{n} + W_n, \tag{7}$$

其中a为常数，而W_n满足

$$E|W_n| = o(n^{-1}). \tag{8}$$

现在证明定理2，以$G_n(x)$记$\sqrt{n}(U_n-\theta)/m\sigma$的分布函数，则由[2]已知有：

$$\|G_n - \Phi\| = O(n^{-\frac{1}{2}}). \tag{9}$$

取 $b=|a|+1$，则

$$F_n(x) - G_n\left(x+\frac{b}{\sqrt{n}}\right) = P\left\{V_n-\theta<\frac{m\sigma x}{\sqrt{n}},\ |W_n|<\frac{m\sigma}{\sqrt{n}}\right\} + P\left\{V_n-\theta<\frac{m\sigma x}{\sqrt{n}},\right.$$

$$\left.|W_n|\geqslant\frac{m\sigma}{\sqrt{n}}\right\} - P\left\{U_n-\theta<\frac{m\sigma x}{\sqrt{n}}+\frac{m\sigma b}{n}\right\} \leqslant P\left\{V_n-\frac{a}{n}-W_n-\theta<\frac{m\sigma x}{\sqrt{n}}\right.$$

$$\left.+\frac{m\sigma}{\sqrt{n}}+\frac{|a|}{n}\right\} - P\left\{U_n-\theta<\frac{m\sigma x}{\sqrt{n}}+\frac{m\sigma b}{\sqrt{n}}\right\} + P\left\{|W_n|\geqslant\frac{m\sigma}{\sqrt{n}}\right\}$$

$$\leqslant P\left\{|W_n|\geqslant\frac{m\sigma}{\sqrt{n}}\right\} \tag{10}$$

由（8），

$$P\left\{|W_n|\geqslant\frac{m\sigma}{\sqrt{n}}\right\} \leqslant \frac{\sqrt{n}}{m\sigma}E|W_n| = \frac{\sqrt{n}}{m\sigma}o(n^{-1}) = o(n^{-\frac{1}{2}}) \tag{11}$$

再注意到$\Phi'(x)$在全直线上有界，则得

$$F_n(x)-\Phi(x) = \left[F_n(x)-G_n\left(x+\frac{b}{\sqrt{n}}\right)\right] + \left[G_n\left(x+\frac{b}{\sqrt{n}}\right)-\Phi\left(x+\frac{b}{\sqrt{n}}\right)\right]$$

$$+\left[\Phi\left(x+\frac{b}{\sqrt{n}}\right)-\Phi(x)\right] \leqslant o\left(n^{-\frac{1}{2}}\right)+O\left(n^{-\frac{1}{2}}\right)+O\left(n^{-\frac{1}{2}}\right) = O\left(n^{-\frac{1}{2}}\right). \tag{12}$$

同理从$F_n(x)-G_n\left(x-\frac{b}{\sqrt{n}}\right)$出发作类似讨论得：

$$F_n(x)-\Phi(x) \geqslant O\left(n^{-\frac{1}{2}}\right). \tag{13}$$

由（12)、(13）两式即得要证明的（4）式。

假如不直接应用文[1]的结论，也可证明定理2，自然证明方法实质上是一致的，为

此先引进一个

引理　设 $X_n = X_{n1} + X_{n2}$，$n = 1, 2, \cdots$ 为一串随机变量，X_n，X_{n1} 的分布函数分别记为 $F_n(x), F_{n1}(x)$，设存在与 n 无关的常数 C，使得

$$\|F_{n1} - \Phi\| = O\left(n^{-\frac{1}{2}}\right), \qquad P\left\{|X_{n2}| \geqslant C n^{-\frac{1}{2}}\right\} = O\left(n^{-\frac{1}{2}}\right), \tag{14}$$

那末：

$$\|F_n - \Phi\| = O(n^{-\frac{1}{2}}). \tag{15}$$

上述引理引自文[3]中引理9．

定理2的证明　由定义

$$\begin{aligned} V_n &= n^{-m} \sum_{\alpha_1, \alpha_2, \cdots, \alpha_m = 1}^{n} \varphi(X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_m}) \\ &= m! n^{-m} \sum_{1 \leqslant \alpha_1 < \cdots \alpha_m \leqslant n} \varphi(X_{\alpha_1}, \cdots, X_{\alpha_m}) + \left[V_n - m! n^{-m} \sum_{1 \leqslant \alpha_1 < \cdots \alpha_m \leqslant n} \varphi(X_{\alpha_1}, \cdots, X_{\alpha_m}) \right] \\ &= \left[1 + O\left(\frac{1}{n}\right) \right] U_n + T_n = U_n + \left[O\left(\frac{1}{n}\right) U_n + T_n \right] \\ &= U_n + Q_n. \end{aligned} \tag{16}$$

这里

$$Q_n = O\left(\frac{1}{n}\right) U_n + T_n, \qquad T_n = V_n - m! n^{-m} \sum_{1 \leqslant \alpha_1 < \cdots \alpha_m \leqslant n} \varphi(X_{\alpha_1}, \cdots, X_{\alpha_m}). \tag{17}$$

显然

$$\begin{aligned} P\left\{\left|O\left(\frac{1}{n}\right) U_n\right| \geqslant C n^{-\frac{1}{2}}\right\} &\leqslant O\left(\frac{1}{n}\right) \cdot n^{\frac{1}{2}} E|U_n| \\ &\leqslant O(n^{-\frac{1}{2}}) \binom{n}{m}^{-1} \binom{n}{m} E|\varphi(X_1, \cdots, X_m)| = O(n^{-\frac{1}{2}}). \end{aligned} \tag{18}$$

而

$$\begin{aligned} P\{|T_n| \geqslant C n^{-\frac{1}{2}}\} &\leqslant C^{-1} n^{\frac{1}{2}} \cdot n^{-m}\left(n^m - m! \binom{n}{m}\right) \max_{1 \leqslant \alpha_1, \cdots, \alpha_m \leqslant m} E|\varphi(X_{\alpha_1}, \cdots, X_{\alpha_m})|. \\ &= C^{-1} n^{-m+\frac{1}{2}} (n^m - n(n-1)\cdots(n-m+1)) \max_{1 \leqslant \alpha_1, \cdots, \alpha_m \leqslant m} E|\varphi(X_{\alpha_1}, \cdots, X_{\alpha_m})| \\ &= O(n^{-\frac{1}{2}}) \max_{1 \leqslant \alpha_1, \cdots, \alpha_m \leqslant m} E|\varphi(X_{\alpha_1}, \cdots, X_{\alpha_m})| = O(n^{-\frac{1}{2}}). \end{aligned} \tag{19}$$

则由(9)，(18)，(19)，再结合到引理，得到(4)式．

参　考　文　献

[1]　陈希孺，U-统计量和von-Mises统计量的极限性质，中国科学6，(1980)，P.522.

[2]　Chan. Y. K. & Wierman. J., *Ann Prob* 6(1978), 186.

[3]　陈希孺，线性模型中误差方差估计的Berry-Esseen界限，中国科学2，(1981)，P.129-140.

A Note on the Berry-Esseen Theorem for von Mises Statistics

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Abstract

Let V_n be a von Mises Statistice with kernel $\varphi(x_1,x_2,\cdots,x_m)$, that is

$$V_n=V_n(X_1,X_2,\cdots,X_n)=n^{-m}\sum_{\alpha_1=1}^{n}\cdots\sum_{\alpha_m=1}^{n}\varphi(X_{\alpha_1},\cdots,X_{\alpha_m}),\ n\geqslant 1$$ where $X_1,X_2,\cdots,X_n,\cdots$ is i. i. d. random variables, $\varphi(X_1,X_2,\cdots,X_m)$ is a symmetric function in its arguments. Suppose that X_1 has finite variance σ^2 and $\theta=EX_1$. Represent distribution function of the variable $\frac{\sqrt{n}(V_n-\theta)}{m\sigma}$ and a standard normal random variable by $F_n(x)$ and $\Phi(x)$ respectively. Then the following theorem has been obtained by [1]:

Suppose that

1) $E[|\varphi(X_{\alpha_1},\cdots,X_{\alpha_m})|]<\infty$, for any $\alpha_j=1,\cdots,n$, $j=1,\cdots,m$ (1)

2) $E[|\varphi(X_1,\cdots,X_m)|]<\infty$, (2)

3) $E[|\varphi(X_1,X_2,X_3,\cdots,X_m)|^{\frac{3}{2}}]<\infty$, (3)

then

$$\|F_n-\Phi\|=\sup_x|F_n(x)-\Phi(x)|=O(n^{-\frac{1}{2}}). \tag{4}$$

The purpose of this paper is to prove the assertion of the theorem can be attained requiring only the conditions (1), (2) are satisfied. And the conclusion thus drawn is in accordance with the supposition mentioned in [1].