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Meromorphic Convex Functions with Negative Coefficients*

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Abstract Let $\sigma_k(a)$ be the class of functions $f(f)=\frac{1}{z}-\sum_{n=1}^{\infty}|a_n|z^n$, regular in the punctured disk $E=\{z:0<|z|<1\}$ and satisfying $\mathrm{Re}(1+zf''(z)/f'(z))<-a$ $(0\le a<1)$ for $z\in E$. In this paper we obtain coefficient inequalities, distortion and closure Theorems for the class $\sigma_k(a)$. Further we obtain the class preserving integral operator of the form

$F(z)=c\int_0^1 u^c f(uz)du$, $(0<u\le 1)$, $c>0$ for the class $\sigma_k(a)$. Conversely when $F\in\sigma_k(a)$ the radius of convexity of f is obtained.

1. Introduction Let Σ denote the class of functions $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n$, regular in the punctured disk $E=\{z:0<|z|<1\}$ with a simple pole at $z=0$. Let Σ_k denote the subclass of Σ consisting of functions $f(z)$ which are convex with respect to the origin, that is, satisfying the condition $\mathrm{Re}(1+zf''(z)/f'(z))<0$ for $z\in E$. And $\Sigma_k(a)$ denote the subclass of Σ_k consisting of functions $f(z)$ which are convex of order a, that is, satisfying the condition $\mathrm{Re}(1+zf''(z)/f'(z))<-a$ for $z\in E$.

Let σ denote the subclass of Σ consisting of functions $f(z)=\frac{1}{z}-\sum_{n=1}^{\infty}|a_n|z^n|$ and let $\sigma_k(a)=\Sigma_k(a)\cap\sigma$. In this paper we obtain coefficient inequalities and distortion Theorems for the class $\sigma_k(a)$. Further it is shown that $\sigma_k(a)$ is closed under convex linear combinations. We also obtain the class preserving integral operators of the form (1) $F(z)=c\int_0^1 u^c f(uz)du$ $(0<u\le 1)$, $c>0$ for the class $\sigma_k(a)$. Conversely when $F\in\sigma_k(a)$, the radius of convexity of f defined

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by (1) is obtained. While considering the integrals we have been able to extend some of the known results of R. M. Goel and N. S. Sohi [2] and S. K. Bajpai [1].

In [4, 5] the authors have examined the meromorphic starlike functions with positive coefficients. Coefficient in equalities, distortion and closure Theorems, radius of convexity and the class preserving integral operators are obtained.

We employ techniques similar to those in [3].

2. Coefficient Inequalities

Theorem I Let $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n$. If $\sum_{n=1}^{\infty}n|a_n|(n+a)\leq 1-a$, then $f\in\Sigma_k(a)$.

Proof It suffices to show that

$$\left|\frac{zf''(z)/f'(z)+2}{zf''(z)/f'(z)+1-(1-2a)}\right|<1 \text{ for } |z|<1.$$

we have

$$\left|\frac{zf''(z)/f'(z)+2}{zf''(z)/f'(z)+1-(1-2a)}\right|=\left|\frac{\sum_{n=i}^{\infty}n(n+1)a_n z^{n+1}}{2(1-a)+\sum_{n=1}^{\infty}(n-1+2a)na_n z^{n+1}}\right|$$

$$\leq\frac{\sum_{n=1}^{\infty}n(n+1)|a_n|}{2(1-a)-\sum_{n=1}^{\infty}(n-1+2a)n|a_n|}.$$

The last expression is bounded above by 1 if

$$\sum_{n=1}^{\infty}n(n+1)|a_n|\leq 2(1-a)-\sum_{n=1}^{\infty}(n-1+2a)n|a_n|,$$

which is equival nt to

$$(2)\qquad \sum_{n=1}^{\infty}n|a_n|(n+a)\leq 1-a.$$

But (2) is true by hypothesis.

For functions in $\sigma_k(a)$ the converse of Theorem 1 is also true.

Theorem 2 A function $f(z)=\frac{1}{z}-\sum_{n=1}^{\infty}|a_n|z^n$ in $\sigma_k(a)$ if and only if $\sum_{n=1}^{\infty}n|a_n|(n+a)\leq 1-a$.

Proof In view of Theorem 1, it suffices to show the only if part. Assume that

$$(3)\quad \mathrm{Re}(1+zf''(z)/f'(z))=\mathrm{Re}\frac{1-\sum_{n=1}^{\infty}n^2|a_n|z^{n+1}}{-1-\sum_{n=1}^{\infty}n|a_n|z^{n+1}}<-a \text{ for } |z|<1.$$

Choose values of z on the real axis. So that $1+zf''(z)/f'(z)$ is real. Upon clearing the denominator in (3) and letting $z\to 1$ through real values, we get

$$1-\sum_{n=1}^{\infty} n^2|a_n|\geq (1+\sum_{n=1}^{\infty} n|a_n|)a.$$

Thus $\sum_{n=1}^{\infty} n|a_n|(n+a)\leq 1-a$.

Corollary 1 If $f\in\sigma_k(a)$ then $|a_n|\leq\frac{1-a}{n(n+a)}$ with equality for functions of the form $f_n(z)=\frac{1}{z}-\frac{1-a}{n(n+a)}z^n$

3. Distortion Theorem

Theorem 3 If $f\in\sigma_k(a)$ then

(4) $\frac{1}{r}-\frac{1-a}{1+a}r\leq|f(z)|\leq\frac{1}{r}+\frac{1-a}{1+a}r \qquad (|z|=r)$

with equality for $f(z)=\frac{1}{z}-\frac{1-a}{1+a}z \quad (z=ir,\ r)$ and

(5) $\frac{1}{r^2}-\frac{1-a}{1+a}\leq|f'(z)|\leq\frac{1}{r^2}+\frac{1-a}{1+a} \quad (|z|=r)$

with equality for $f(z)=\frac{1}{z}-\frac{1-a}{1+a}z \quad (z=\pm r,\ \pm ir)$.

Proof Let $f(z)=\frac{1}{z}-\sum_{n=1}^{\infty}|a_n|z^n$. Then it follows from Theorem 2 that $\sum_{n=1}^{\infty}|a_n|\leq\frac{1-a}{1+a}$. Hence

$$|f(z)|\leq\frac{1}{r}+\sum_{n=1}^{\infty}|a_n|r^n\leq\frac{1}{r}+r\sum_{n=1}^{\infty}|a_n|\leq\frac{1}{r}+\frac{1-a}{1+a}r$$

and

$$|f(z)|\geq\frac{1}{r}-\sum_{n=1}^{\infty}|a_n|r^n\geq\frac{1}{r}-r\sum_{n=1}^{\infty}|a_n|\geq\frac{1}{r}-\frac{1-a}{1+a}r.$$

Thus (4) follows. Further from Theorem 2 it follows that

$$\sum_{n=1}^{\infty}n|a_n|\leq\frac{1-a}{1+a}.$$

Hence

$$|f'(z)|\leq\frac{1}{r^2}+\sum_{n=1}^{\infty}n|a_n|r^{n-1}\leq\frac{1}{r^2}+\sum_{n=1}^{\infty}n|a_n|\leq\frac{1}{r^2}+\frac{1-a}{1+a}$$

and

$$|f'(z)| \geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n| r^{n-1} \geq \frac{1}{r^2} - \sum_{n=1}^{\infty} n|a_n| \geq \frac{1}{r^2} - \frac{1-a}{1+a} .$$

Thus (5) follows.

4. Closure Theorem.

In this section we shall show that the class $\sigma_k(a)$ is closed under convex linear combinations.

Theorem 4 Let $f_0(z) = \frac{1}{z}$ and $f_n(z) = \frac{1}{z} - \frac{(1-a)}{n(n+a)} z^n$ for $n = 1, 2, \cdots$

Then $f \in \sigma_k(a)$ if and only if $f(z)$ can be expressed in the form

$f(z) = \sum_{n}^{\infty} \lambda_n f_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$

Proof Suppose $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \frac{1}{z} - \sum_{n=1}^{\infty} \lambda_n \frac{1-a}{n(n+a)} z^n$, then

$$\sum_{n=1}^{\infty} \lambda_n \frac{(1-a)n(n+a)}{n(n+a)} = (1-a) \sum_{n=1}^{\infty} \lambda_n = (1-a)(1-\lambda_0) \leq 1-a.$$

Hence $f(z) \in \sigma_k(a)$.

Conversely suppose $f(z) \in \sigma_k(a)$. Since from corollary 1, we have

$$|a_n| \leq \frac{1-a}{n(n+a)}, \quad n = 1, 2, \cdots$$

taking $\lambda_n = \frac{n(n+a)}{1-a} |a_n|$, $n = 1, 2, \cdots$ and $\lambda_0 = 1 - \sum_{n=0}^{\infty} \lambda_n$, we have

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) .$$

5. Integral Operators.

Theorem 5 If $f(z) = \frac{1}{z} - \sum_{n=1}^{\infty} |a_n| z^n$ is in $\sigma_k(a)$, then

$$F(z) = c\int_0^1 u^c f(uz) du = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{c}{c+n+1} |a_n| z^n, \quad 0 < u \leq 1, \ c > 0$$

belongs to $\sigma_k\left(\frac{1+a+ac}{1+a+c}\right)$ for $F(z) \neq 0$ in $0 < |z| < 1$. The result is sharp for $f(z) = \frac{1}{z} - \frac{1-a}{1+a} z$.

Proof Suppose $f(z) \in \sigma_k(a)$ then $\sum_{n=1}^{\infty} n|a_n|(n+a) \leq 1-a$. In view of Theorem 2 we shall find the largest β for which $\sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta} \frac{c}{c+n+1} |a_n| \leq 1$.

It suffices to find the range of values of β for which

$$\frac{c(n+\beta)}{(1-\beta)(c+n+1)} \leq \frac{n+a}{1-a} \quad \text{for each } n .$$

From the above inequlity we obtain

$$\beta \leq \frac{n^2+n(1+a+ac)+a(c+1)}{n^2+n(1+a+c)+c+a}.$$

For each a and c fixed, let $F(n)=\frac{n^2+n(1+a+ac)+a(c+1)}{n^2+n(1+a+c)+c+a}$. Then $F(n+1)$

$$-F(n)=\frac{c(1-a)(n+1)(n+2)}{\{n^2+n(1+a+c)+c+a\}\{(n+1)^2+(n+1)(1+a+c)+c+a\}}>0 \text{ for}$$

each n.

Hence $F(n)$ is an increasing function of n. Since $F(1)=\frac{1+a+ac}{1+a+c}$ the result follows.

Corollary 2 If $f(z)=\frac{1}{z}-\sum_{n=1}^{\infty}|a_n|z^n$ satisfies $\mathrm{Re}(1+zf''(z)/f'(z))<1/(1+c)$ for $|z|<1$, then $F(z)=c\int_0^1 u^c f(uz)\mathrm{d}u$, $(0<u\leq 1)$, $c>0$ is convex for $F(z)\neq 0$ in $0<|z|<1$.

Remark The above corollary extends a result of Goel and Sohi [2] for functions with negative coefficients. In fact Goel and Sohi have proved that if $f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_n z^n$ satisfies $\mathrm{Re}(1+zf''(z)/f'(z))<1/2(c+1)$, for $|z|<1$ then $F(z)=c\int_0^1 u^c f(uz)\mathrm{d}u$, $(0<u\leq 1)$, $c>0$ is convex for $F(z)\neq 0$ in $0<|z|<1$.

Putting $c=1$ in corollary 2 we obtain the following result.

Corollary 3 If $f(z)=\frac{1}{z}-\sum_{n=1}^{\infty}|a_n|z^n$ satisfies $\mathrm{Re}(1+zf''(z)/f'(z))<1/2$ for $|z|<1$, then $F(z)=\int_0^1 uf(uz)\mathrm{d}u$, $(0<u\leq 1)$ is convex for $F(z)\neq 0$ in $0<|z|<1$.

Remark The above corollary extends a result of Bajpai [1] for functions with negative coefficients. In fact Bajpai has proved that if $f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_n z^n$ is convex then so is $F(z)=z^{-2}\int_0^z tf(t)\mathrm{d}t$.

Theorem 6 Let $F(z)=\frac{1}{z}-\sum_{n=1}^{\infty}|a_n|z^n$ be in $\sigma_k(a)$ and

$$f(z)=\frac{1}{c}[(c+1)F(z)+zF'(z)]=\frac{1}{z}\pm\sum_{n=1}^{\infty}\frac{c+n+1}{c}|a_n|z^n,\quad c>0.$$

Then $\mathrm{Re}(1+zf''(z)/f'(z))<-\beta$, $(0\leq\beta<1)$ for $0<|z|<r(a,\beta)$ where

$$r(\alpha,\ \beta)=\inf\left(\frac{c(1-\beta)(n+\alpha)}{(1-\alpha)(c+n+1)(n+\beta)}\right)^{\frac{1}{n+1}},\quad n=1,\ 2,\ \cdots.$$

The result is sharp for $F(z)=\frac{1}{z}-\frac{1-\alpha}{n(n+\alpha)}z^n$ for some n.

Proof It suffices to show that

$$\left|\frac{zf''(z)/f'(z)+2}{zf''(z)/f'(z)+2\beta}\right|<1 \quad \text{for} \quad 0<|z|<r(\alpha,\beta).$$

We have

$$\left|\frac{zf''(z)/f'(z)+2}{zf''(z)/f'(z)+2\beta}\right|\leq\frac{\sum_{n=1}n(n+1)\frac{c+n+1}{c}|a_n||z|^{n+1}}{2(1-\beta)-\sum_{n=1}^{\infty}(n-1+2\beta)n\frac{c+n+1}{c}|a_n||z|^{n+1}}.$$

The last expression is bounded above by 1 if

$$(6)\quad \sum_{n=1}^{\infty}\frac{n(n+\beta)(c+n+1)}{(1-\beta)c}|a_n||z|^{n+1}\leq 1.$$

From Theorem (2) we have $\sum_{n=1}^{\infty}\frac{n(n+\alpha)}{1-\alpha}|a_n|\leq 1$. Hecne (6) is satisfied if

$$(7)\quad \frac{(n+\beta)(c+n+1)}{(1-\beta)c}|z|^{n+1}\leq\frac{n+\alpha}{1-\alpha},\quad n=1,\ 2,\ \cdots.$$

Solving (7) for $|z|$ we obtain

$$(8)\quad |z|\leq\left(\frac{c(1-\beta)(n+\alpha)}{(1-\alpha)(n+\beta)(c+n+1)}\right)^{\frac{1}{n+1}},\quad n=1,\ 2,\ \cdots.$$

Writing $|z|=r(\alpha,\ \beta)$ in (8), the result follows.

We have $r(\alpha,\alpha)=\inf\left(\frac{c}{c+n+1}\right)^{\frac{1}{n+1}}=\sqrt{c/c+2},\ n=1,2,\cdots$

Remark The above Theorem extends a result of Goel and Sohi [2] for functions with negative coefficients. In fact Goel and Sohi have proved that:

If $F(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n$ is convex and $f(z)=\frac{1}{c}[(c+1)F(z)+zF'(z)]$, $c>0$, then $f(z)$ is convex in $0<|z|<\sqrt{c/c+2}$. The rcsult is sharp.

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