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Fixed Points of Nonexpansive Type Mappings in Topological Spaces *

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1 Introduction

Recently, Penot [1] and Kirk [2] obtained the abstract versions of Kirk's fixed point theorem of [3] for nonexpansive mappings, respectively. Gillespie and Williams [4, 5] replaced the reflexive and normal structure used in Kirk [3] and Kannan [6], by uniformly normal structure to obtain the fixed point theorems for nonexpansive and Kannan mappings. Kaukich [7] also extended the result of [3] to the generalized nonexpansive mappings.

In this paper, we obtain several new abstract fixed point theorems for generalized nonexpansive and Kannan mappings in Hausdorff uniform topological spaces which improve and generalize some main results of [4, 5, 8].

2 Notation and Definition

Let (X, τ) be a Hausdorff uniform topological space where the uniform topology τ is generated by the family $\mathscr{P}$ of pseudo-metrics on X. A sequence $\{x_n\} \subset X$ is said to be τ-cauchy if and only if it is cauchy under the topology generated by the pseudo-metric p for each $p \in \mathscr{P}$. We say that the sequence $\{x_n\}$ τ-converges to u if and only if it converges to u under the topology generated by p for each $p \in \mathscr{P}$. The space (X, τ) is said to be complete if each τ-cauchy sequence $\{x_n\}$ converges to a point x in X under the topology τ. Also note that for any $x, y \in X$ $x \neq y$ if and only if there exists $p \in \mathscr{P}$ such that $p(x, y) > 0$. A set $F \subset X$ is said to be bounded if for each $p \in \mathscr{P}$ there is a real number $M_p > 0$ such that $\delta_p(F) = \sup\{p(x, y): x, y \in F\} < M_p$, where $\delta_p(F)$ is the p-diameter of F.

For each $p \in \mathscr{P}$ and any bounded subset $F \subset X$ let

$$r_x(F, p) = \sup\{p(x, y): y \in F\}. \text{ and}$$

$$r(F, p) = \inf\{r_x(F, p): x \in F\}.$$

Definition 1 A family $\mathscr{F}$ of subsets of (X, τ) is said to be compact if each subfamily of $\mathscr{F}$ which has the finite intersection property has nonvoid

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intersection.

Definition 2 Let f be a selfmapping of (X, τ) and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semicontinuous function from the right and satisfies $\varphi(t)>t$, $\forall t>0$. The space (X, τ) is said to have strong normal structure if for each $p\in\mathscr{P}$

$$\varphi(r(F, p))\leq\delta_p(F)$$

whenever F is a bounded closed f invariant subset of X.

Definition 3 Let f be a selfmapping of (X, τ). f is said to be a generalized nonexpansive mapping if for each $p\in\mathscr{P}$ and all $x, y\in X$

(A) $\quad p(fx, fy)\leq a\sup\{p(x, u):u\in\{f^n x\}_{n\geq 0}\cup\{f^n y\}_{n\geq 0}\}+b\,p(y, fx)$

where $a, b\geq 0$, $b\neq 1$ and $a+b\leq 1$.

Definition 4 Let f be a selfmapping of (X, τ). f is said to be a generalized Kannan mapping if for each $p\in\mathscr{P}$ there exists $a_p:X\times X\to[0,\infty)$ satisfying $a_p(x, y)+a_p(y, x)\leq 1$ and $\inf\limits_{x, y\in X} a_p(x, y)>0$ such that for all $x, y\in X$

$$p(fx, fy)\leq a_p(x, y)\,p(x, fx)+a_p(y, x)\,p(y, fy).$$

3 Main Results

Lemma 1 Let (X, τ) be a complete Hausdorff uniform topological space Let $\{A_n\}$ be a decreasing sequence of nonempty closed subsets A_n of X such that

$$\lim_{n\to\infty}\delta_p(A_n)=0$$

for all $p\in\mathscr{P}$. Then

$$\bigcap_n A_n=\{x^*\}, \text{ where } x^*\in X.$$

Proof For each n choose an element $x_n\in A_n$. since $A_n\subset A_m$ if $n\geq m$, then $x_n\in A_m$ for all $n\geq m$. Given $\varepsilon>0$, for every $p\in\mathscr{P}$ there exists a $n_p>0$ such that $\delta_p(A_m)\leq\delta_p(A_{n_p})\leq\varepsilon$ whenever $m\geq n_p$. Thus if $m, n\geq n_p$, $p(x_m, x_n)\leq\delta(A_{n_p})\leq\varepsilon$ and so $\{x_n\}$ is a p-cauchy sequence for each $p\in\mathscr{P}$. Therefore $\{x_n\}$ is also a τ-cauchy sequence and so $\{x_n\}$ converges to an element x^* in X, since X is complete. Moreover, for each fixed n, $\{x_{n+i}\}$ converges to x^* as $i\to\infty$. Since $x_{n+i}\in A_n$ for each i and since A_n is closed, we conclude $x^*\in\bar{A}_n=A_n$. Thus $x^*\in\bigcap_n A_n=A$. It follows that for each $p\in\mathscr{P}$ $\delta(A)=0$, which implies that A contains at most a single poine, since X is Hausdorff. Therefore, $\bigcap_n A_n=\{x^*\}$.

Theorem 1 Let (X, τ) be a bounded complete Hausdorff uniform topological space, $f:X\to X$ is a generalized nonexpansive mapping satisfying the condition (A). If X has strongly normal structure about f, Then f has a fixed point in X.

Proof Let $\mathscr{F}=\{F:F$ is a nonempty f-invariant closed subset of $X\}$. For each fixed $x\in X$, let $L=L(x)=\bigcap\{F\in\mathscr{F}:x\in F\}$ and let $L_1=\mathrm{cl}\{f(L)\cup\{x\}\}$ (clA denote the closure of A). we see easily that $L_1\subset L$ and $f(L_1)\subset f(L)\subset L_1$ and so $L = L_1$. Let

$$L_2 = \{y \in L : r_y(L, p) \leq r_x(L, p) \text{ far all } p \in \mathscr{P}\}.$$

Then $L_2 \neq \phi$, since $x \in L_2$. From the definition of $r_y(L, p)$ and the continuity of each $p \in \mathscr{P}$, it follows that L_2 is a closed set. Given $\varepsilon > 0$, for any $w \in L$ and each $p \in \mathscr{P}$ there is a $z_p \in f(L) \cup \{x\}$ such that $p(w, z_p) \leq \varepsilon$, since $L = L_1 = \mathrm{cl}\{f(L) \cup \{x\}\}$. Also if $z_p = x$, then for each $y \in L_2$

$$p(fy, w) \leq p(fy, z_p) + p(z_p, w)$$
$$\leq p(fy, x) + \varepsilon \leq r_x(L, p) + \varepsilon,$$

since $f(L_2) \subset f(L) \subset L$. By the arbitrariness of ε and $w \in L$, we obtain

$$r_{fy}(L, p) \leq r_x(L, p), \quad \forall p \in \mathscr{P}.$$

If $z_p \in f(L)$, then there is a $z_p^* \in L$ such that $z_p = fz_p^*$ and for each $y \in L_2 \subset L$,

$$p(fy, w) \leq p(fy, fz_p^*) + p(z_p, w)$$
$$\leq a \sup\{p(y, u) : u \in \{f^n y\}_{n \geq 0} \cup \{f^n z_p^*\}_{n \geq 0}\} + b\, p(z_p^*, fy) + \varepsilon.$$

since $y, z_p^* \in L$ and $f(L) \subset L$, we have $\{f^n y\}_{n \geq 0} \cup \{f^n z_p^*\}_{n \geq 0} \subset L$. Therefore we obtain

$$p(fy, w) \leq a\, r_y(L, p) + b\, r_{fy}(L, p) + \varepsilon \leq a\, r_x(L, p) + b\, r_{fy}(L, p) + \varepsilon,$$

since $y \in L_2$. By the arbtrariness of $w \in L$ and ε, we deduce

$$r_{fy}(L, p) \leq \frac{a}{1-b} r_x(L, p) \leq r_x(L, p), \quad \forall p \in \mathscr{P}$$

and hence $fy \in L_2$ and $f(L_2) \subset L_2$ which in turn implies $L = L_2 = L(x) = \{y \in L(x) : r_y(L(x), p) \leq r_x(L(x), p)$, for all $p \in \mathscr{P}\}$ and $\delta_p(L(x)) = r_x(L(x), p)$, $\forall p \in \mathscr{P}$.

Now consider $x_0 \in X$. By the definitions of $\delta_p(L(x))$ and $r(L(x), p)$, there exists a sequence $\{x_n\}_{n \geq 0}$ such that

$$x_{n+1} \in L(x_n),$$
$$r_{x_{n+1}}(L(x_n), p) \leq \frac{1}{2}[r(L(x_n), p) + \delta_p(L(x_n))], \quad \forall n = 0, 1, \cdots \text{and } p \in \mathscr{P}.$$

Thus we have $L(x_{n+1}) \subset L(x_n)$ and

$$\delta_p(L(x_{n+1})) = r_{x_{n+1}}(L(x_{n+1}), p) \leq r_{x_{n+1}}(L(x_n), p)$$
$$\leq \frac{1}{2}[r(L(x_n), p) + \delta_p(L(x_n))], \forall n = 0, 1, \cdots \text{and } p \in \mathscr{P} \quad (1)$$

since $\{\delta_p(L(x_n))\}_{n \geq 0}$ and $\{r(L(x_n), p)\}_{n \geq 0}$ are nonnegative decreasing sequeces, let $\lim_{n \to \infty} \delta_p(L(x_n)) = \alpha_p$ and let $\lim_{n \to \infty} r(L(x_n), p) = \beta_p$. obviously, $\beta_p \leq \alpha_p$. On the other hand, putting $n \to \infty$ in (1) we have $\alpha_p \leq \beta_p$ and so $\alpha_p = \beta_p$. Also since X has strongly normal structure about f, we get

$$\varphi(r(L(x_n), p)) \leq \delta_p(L(x_n)), \ \forall n = 0, 1, \cdots \text{and } p \in \mathscr{P}. \quad (2)$$

since φ is lower semicontinuous from the right, letting $n \to \infty$ in (2) we obtain $\varphi(\alpha_p) \leq \alpha_p$ which yield $\alpha_p = 0$. $\forall p \in \mathscr{P}$. Using Lemma 1 we deduce that $\bigcap_{n=0}^{\infty} L(x_n) = \{x^*\}$ and hence $x^* = fx^*$ This completes the proof of Theorem 1.

As an immediate consequence of Theorem 1, we have the following

Corollary 1 Let (X, d) be a bounded complete metric space, and $f: X \to X$ is such that for all $x, y \in X$

$$d(fx, fy) \leq a \sup\{d(x, u): u \in \{f^n x\}_{n\geq 0} \cup \{f^n y\}_{n\geq 0}\} + bd(y, fx),$$

where $a, b \geq 0$, $b \neq 1$, and $a+b \leq 1$. If X has strongly normal structure about f, Then f has a fixed point in X.

Proof The desired conclusion follows from Theorem 1.

Remark 1 The corollary 1 is an improvement of theorem of [4] for the cace of metric spaces.

Theorem 2 Let (X, τ) be a bounded complete Hausdorff uniform topological space and $f: X \to X$ a generalized Kannan mapping. suppose that there exists a nondecreasing function $\varphi: [0, \infty) \to [0, \infty)$ satisfying $\lim_{n\to\infty} \varphi^n(t) = 0$, $\forall t > 0$, where φ^n denotes the n-th iterate of φ. If for each f-invariant closed subset E of X there is a $x_0 \in E$ such that

$$p(x_0, fx_0) \leq \varphi(\delta_p(E))$$

for all $p \in \mathscr{P}$, then f has a unique fixed point in X.

Proof We have $E = \{x \in X: p(x, fx) \leq \varphi(\delta_p(X)), \forall p \in \mathscr{P}\} \neq \phi$. Let $E_1 = cl(f(E))$, then for any given $\varepsilon > 0$, $y \in E_1$ and for each $p \in \mathscr{P}$, there is a $z_p \in E$ such that $p(y, fz_p) \leq \varepsilon$. Therefore, we have

$$p(y, fy) \leq p(y, fz_p) + p(fz_p, fy) \leq \varepsilon + a_p(z_p, y)p(z_p, fz_p) +$$
$$+ a_p(y, z_p)p(y, fy) \leq \varepsilon + a_p(z_p, y)\varphi(\delta_p(X)) + (1 - a_p(z_p, y))p(y, fy),$$

and so

$$a_p(z_p, y)p(y, fy) \leq \varepsilon + a_p(z_p, y)\varphi(\delta_p(X)).$$

By the arbitrariness of ε and $\inf_{x, y \in X} a_p(x, y) > 0$, we have

$$p(y, fy) \leq \varphi(\delta_p(X)), \quad \forall p \in \mathscr{P};$$

which in turn implies $y \in E$ and $E_1 \subset E$. Thus $f(E_1) \subset f(E) \subset E_1$ and $\delta_p(E_1) = \delta_p(f(E_1))$, $\forall p \in \mathscr{P}$.

On the other hand, for each $p \in \mathscr{P}$ and for all $x, y \in E$, we have

$$p(fx, fy) \leq a_p(x, y)p(x, fx) + a_p(y, x)p(y, fy) \leq \varphi(\delta_p(X))$$

which yields

$$\delta_p(E_1) = \delta_p(f(E)) \leq \varphi(\delta_p(X)), \quad \forall p \in \mathscr{P}.$$

By induction, there exists a sequence $\{E_n\}$ of nonempty f-invariant closed subsets of X which satisfies:

$$X \supset E_1 \supset E_2 \supset \cdots$$

and

$$\delta_p(E_{n+1}) \leq \varphi(\delta_p(E_n)) \leq \cdots \leq \varphi^{n+1}(\delta_p(X)), \quad \forall p \in \mathscr{P}. \qquad (3)$$

From (3) and the assumption of φ it follow that

$$\lim_{n\to\infty}\delta_p(E_n)=0 \quad \forall p\in\mathscr{P}.$$

Lemma 1 implies that $\bigcap_{n=1}^{\infty} E_n=\{x^*\}$ and hence $fx^*=x^*$.

Now suppose that y^* is a second fixed point of f. Then for each $p\in\mathscr{P}$.

$$p(x^*,y^*)=p(fx^*,fy^*)\leq a_p(x^*,y^*)p(x^*,fx^*)$$
$$+a_p(y^*,x^*)p(y^*fy^*)=0,$$

which implies $x^*=y^*$. Hence x^* is a unique fixed point of f in X.

Remark 2 Clearly, Theorem 2 improve and generalize Theorem 1 of Gillespie and Williams [5] from several aspects.

Theorem 3 Let (X,τ) be a bounded Hausdorff uniform space and $f:X\to X$ a generalized Kannan mapping. If each family of nonempty f-invariant closed subsets of X is compact. Then for each $p\in\mathscr{P}$ there exists a $x_p\in X$ such that

$$p(x_p,fx_p)=\inf\{p(x,fx):x\in X\}.$$

Proof For each $p\in\mathscr{P}$, let $K_{p,r}=\{y\in X:p(y,fy)\leq r\}$ and let $I_p=\{r\in(0,\infty):K_{p,r}\neq\phi\}$. Then $I_p\neq\phi$, since X is bounded. Let $H_{p,r}=\mathrm{cl}(f(K_{p,r}))$, we have $K_{p,r}\subset K_{p,s}$ for any $r,s\in I_p$, $r<s$, and so $H_{p,r}\subset H_{p,s}$ for any $r,s\in I_p$, $r<s$.

Obviously the family $\mathscr{F}_p=\{H_{p,r}:r\in I_p\}$ has the finite intersection property.

Now we shall show that each $H_{p,r}$ is f-invariant. In fact, given $\varepsilon>0$, for any $r\in I_p$ and $y\in H_{p,r}$ there exists a $x\in K_{p,r}$ such that $p(y,fx)\leq\varepsilon$. Therefore we have

$$p(y,fy)\leq p(y,fx)+p(fx,fy)\leq\varepsilon+a_p(x,y)p(x,fx)$$
$$+a_p(y,x)p(y,fy)\leq\varepsilon+a_p(x,y)\cdot r+(1-a_p(x,y))p(y,fy),$$

and so

$$a_p(x,y)p(y,fy)\leq\varepsilon+a_p(x,y)\cdot r.$$

By the arbitrariness of ε and $\inf_{x,y\in X}a_p(x,y)>0$, we obtain $p(y,fy)\leq r$, which implies $y\in K_{p,r}$ and $H_{p,r}\subset K_{p,r}$. This in turn implies $f(H_{p,r})\subset f(K_{p,r})\subset H_{p,r}$, and hence $H_{p,r}$ is f invariant for all $r\in I_p$. Because of the compactness of $\mathscr{F}_p$, $\bigcap_{r\in I_p}H_{p,r}\neq\phi$. Thus, for each $p\in\mathscr{P}$ there is a $x_p\in X$ such that

$$p(x_p,fx_p)=\inf\{p(x,fx):x\in X\}$$

Corollary 2 Let (X,d) be a bounded metric space and $f:X\to X$ is such that for all $x,y\in X$

$$d(fx,fy)\leq a(x,y)d(x,fx)+a(y,x)d(y,fy), \qquad (4)$$

where $a:X\times X\to[0,\infty)$ satisfies $a(x,y)+a(y,x)\leq 1$ and $\inf_{x,y\in X}a(x,y)>0$. If each family of nonempty f-invariant closed subsets of X is compact. Then there is a $x^*\in X$ such that

$$d(x^*, fx^*) = \inf\{d(x, fx) : x \in X\}.$$

Remark 3 It is easy to check that Theorem 1 of Wong [8] is a very special case of corollary 2.

Theorem 4 Let (X, τ) be a bounded Hausdorff uniform topological space and f: $X \to X$ a generalized Kannan mapping. If each family of nonempty f-invariant closed subsets of X is compact, then the following conditions are equivalent:

(a) f has a unique fixed point in X,

(b) For each $p \in \mathscr{P}$, $\inf\{p(x, fx) : x \in X\} = 0$,

(c) for each $p \in \mathscr{P}$ and $x \in X$ with $p(x, fx) > 0$, there is a $y \in X$ such that $p(x, fx) > p(y, fy)$.

Proof It is clear that (a)⇒(b) and (b)⇔(c), we only have need to prove (b)⇒(a). From (b) and Theorem 3 it follows that the set

$$A_p = \{x \in X: p(x, fx) = 0\} \neq \phi$$

for each $p \in \mathscr{P}$. Let $B_p = \mathrm{cl}(f(A_p))$. Using same argument as in the proof of Theorem 3, we can show $f(B_p) \subset B_p$.

We now prove that the family $\{B_p : p \in \mathscr{P}\}$ has the finite intensection property. Let $\mathscr{U}$ be the subbase of the uniformity generated by $\mathscr{P}$ on X. Let $p_i \in \mathscr{P}$, $i = 1, 2, \cdots, n$ and let $U_i \in \mathscr{U}$ be such that p_i is gauge of U_i, $i = 1, 2, \cdots, n$. Let $U = \bigcap_{i=1}^{n} U_i$. Then there is a $V \in \mathscr{U}$ such that V U. Let p_V and p_U be the gauges of V and U respectively. since $V \subset U \subset U_i$, $i = 1, 2, \cdots, n$, we have $p_V \geq p_U \geq p_i$, $i = 1, 2, \cdots, n$, By (b) and Theorem 3 there is $y \in A_{p_V}$ such that $p_V(y, fy) = 0$. Hence

$$0 = p_V(y, fy) \geq p_U(y, fy) \geq p_i(y, fy) \geq 0, \quad i = 1, 2, \cdots n.$$

Therefore $y \in A_{p_i}$ and hence $A_{p_V} \subset A_{p_i}$ which imply $B_{p_V} \subset B_{p_i}$, $i = 1, 2, \cdots, n$. Thus $fy \in \bigcap_{i=1}^{n} B_{p_i}$. By the compactness of the family $\{B_p : p \in \mathscr{P}\}$, we have $\bigcap_{p \in \mathscr{P}} B_p \neq \phi$. Let $x^* \in \bigcap_{p \in \mathscr{P}} B_p$. Then $p(x^*, fx^*) = 0$ for all $p \in \mathscr{P}$. This means that x^* is a fixed point of f in X. The uniqueness of x^* is clear.

Corollary 3 Let (X, d) be a bounded metric space and $f: X \to X$ satisfies (4) for all $x, y \in X$. If each family of nonempty f-invariant closed subsets of X is compact. Then the following conditions are equivalent:

(a) f has a unique fixed point in X.

(b) $\inf\{d(x, fx) : x \in X\} = 0$,

(c) for each $x \in X$ with $d(x, fx) > 0$ there is a $y \in X$ such that $d(x, fx) > d(y, fy)$.

Remark 4 Theorem 2 of Wong [8] is a special case of corollary 3.

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(from p48)

Let $P^{(m)}(x)=\psi(x, m)/\psi(x, n+1)$, then

$$t_m=\sum_{n=0}^{m} C_{m,n} s_n \qquad (3)$$

$C_{m,n}=c_{n+1}\binom{m}{n}\Delta^{m-n}(p^{(m)}(n)\mu_n)$.

Theorem In order that the transformation (3) should be regular, it is necessary and sufficient that for every m sequence $(P^{(m)}(n)\mu_n)$ should be the difference of two totally monotone sequences, that

$$\Delta^m(P^{(m)}(0)\mu_0)\to 0 \quad (m\to\infty)$$

and that $\mu_0=1$.

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