J.Math.Res. & Exposition.1987年. No.1.

The Global Solution for One Class of the System of LS Nonlinear Wave Interaction*

Guo Boling (郭柏灵)

(P.O.Box 8009, Beijing)

1. Introdution

Recently Djordjevie and Redekopp [1] found that the evolution equations describing the resonance interaction between the long wave and the short wave can be written as

$$iS_t + \lambda S_{xx} = LS \tag{1.1}$$

$$L_t = -\alpha\,(|S|^2)_x \tag{1.2}$$

In the above equations, S is the envelope of the short wave, while L is amplitude of long wave and is real. λ and α are positive constants. As point out in [1], the physical significance of Eqa (1.1) and (1.2) is such that the dispersion of the short wave is balanced by nonlinear Interaction of the long wave with short wave, while the evolution of the long wave is driven by the self-interaction of the short wave. These equations also appear in an analysis of internal wave [2], as well as Rossby Waves. In plasma physics [3] similar equations which describe the resonance between highfrequency electron plasma oscillations and associated low-frequency ion density perturbations. Benney [4] presented a general theory for the interactions between short waves and long waves.

In this paper, we prove that the existence and uniqueness of the global solution for the initial value problem and periodic initial value problem of the equations (1.1) (1.2).

2. The Interal Estimates

For convenience we consider the following system of the equations

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} - \alpha n\varepsilon = 0 & (2.1) \\ n_t + \beta\,(|\varepsilon|^2)_x = 0 & (2.2) \end{cases}$$

with the initial conditions

$$\varepsilon|_{t=0} = \varepsilon_0(x), \quad n|_{t=0} = n_0(x), \quad -\infty < x < \infty, \tag{2.3}$$

---

* *Received Apr.27.1983.*

where $\varepsilon(x,t)$ is a complex valued unknown function, $n(x,t)$ is a real valued unknown function. a, β are real constants, $i=\sqrt{-1}$, we always assume that the functions ε, n and some of their derivatives tend to zero as $|x| \to \infty$, and adopt the notation and convention as in [5].

Lemma 1 If $\varepsilon_0(x) \in L_2$, then for the solution of the problem (2.1) (2.2) (2.3), we have

$$\|\varepsilon(\cdot,t)\|_{L_2} = \|\varepsilon_0(x)\|_{L_2} \tag{2.4}$$

Proof Multiplying (2.1) by $\bar{\varepsilon}$, and taking the inner product it follows

$$(i\varepsilon_t, \varepsilon) + (\varepsilon_{xx}, \varepsilon) - a(n\varepsilon, \varepsilon) = 0.$$

Taking the imaginary part of the above inequality, We derive

$$(\varepsilon_{xx}, \varepsilon) = -\|\varepsilon_x\|^2_{L_2}, \quad \frac{d}{dt}\|\varepsilon\|^2_{L_2} = 0, \text{ i.e. } \|\varepsilon(\cdot,t)\|_{L_2} = \varepsilon_0(x)\|_{L_2}.$$

Lemma 2 Suppose that the following conditions

(i) $\varepsilon_0(x) \in H^1$, (ii) $n_0(x) \in L_2$,

are satisfied, then we have

$$E(t) \equiv \|\varepsilon_x(t)\|^2_{L_2} + a\int n(\cdot,t)|\varepsilon(\cdot,t)|^2 dx = E(0) \tag{2.5}$$

Proof Multiplying (2.1) by $\bar{\varepsilon}_t$, and taking the inner product, it follows $(i\varepsilon_t, \varepsilon_t) + (\varepsilon_{xx}, \varepsilon_t) - (n\varepsilon, \varepsilon_t) = 0$, Taking the real part of the above equation, it yields

$$-\frac{1}{2}\frac{d}{dt}\|\varepsilon_x\|^2_{L2} - \frac{a}{2}\int n|\varepsilon|^2_t dx = 0.$$

Since

$$\int n|\varepsilon|^2_t dx = \frac{d}{dt}\int n|\varepsilon|^2 dx - \int n_t|\varepsilon|^2 dx, \quad \int n_t|\varepsilon|^2 dx = -\beta\int(|\varepsilon|^2)_x|\varepsilon|^2 dx = 0$$

hence we obtain (2.5). $\varepsilon_0(x) \in H^1$, so $\varepsilon_0(x) \in L_4$, it follows

$$\int n_0|\varepsilon_0|^2 dx \le \frac{1}{2}\|n_0(x)\|^2_{L_2} + \frac{1}{2}\|\varepsilon_0\|^4_{L_4} < \infty.$$

Hence E(0) exists.

Lemma 3 If the conditions of lemma 2 are satisfied, then we have

$$E_1(t) = \int n^2(x,t)dx + \frac{\beta}{2a}\int i(\varepsilon\bar{\varepsilon}_x - \bar{\varepsilon}\varepsilon_x)dx = E_1(0) \tag{2.6}$$

Proof Since

$$\frac{d}{dt}\int n^2 dx = \int nn_t dx = \int n\beta(-\varepsilon\bar{\varepsilon})_x dx = -\beta\int(n\varepsilon\bar{\varepsilon}_x + n\bar{\varepsilon}\varepsilon_x)dx = (-\beta/a)\int[(i\varepsilon_t + \varepsilon_{xx})\bar{\varepsilon}_x + (-i\bar{\varepsilon}_t + \bar{\varepsilon}_{xx})\varepsilon_x]dx = (-\beta/a)\int[i\varepsilon_t\bar{\varepsilon}_x - i\bar{\varepsilon}_t\varepsilon_x]dx$$

$$\frac{d}{dt}(\beta/a)\int[i\varepsilon\bar{\varepsilon}_x - i\bar{\varepsilon}\varepsilon_x]dx = (\beta/a)\int[i\varepsilon_t\bar{\varepsilon}_x + i\varepsilon\bar{\varepsilon}_{xt} - i\bar{\varepsilon}_t\varepsilon_x - i\bar{\varepsilon}\varepsilon_{xt}]dx$$
$$= (\beta/a)\int[i\varepsilon_t\bar{\varepsilon}_x - i\varepsilon_x\bar{\varepsilon}_t - i\bar{\varepsilon}_t\varepsilon_x + i\bar{\varepsilon}_x\varepsilon_t]dx = \frac{2\beta}{a}\int[i\varepsilon_t\bar{\varepsilon}_x - i\varepsilon_x\bar{\varepsilon}_t]dx,$$

hence

$$\frac{d}{dx}\int n^2 dx + \frac{\beta}{2a}\frac{d}{dt}\int[i\varepsilon\bar{\varepsilon}_x - i\bar{\varepsilon}\varepsilon_x]dx = 0, \text{ it follows (2.6).}$$

Lemma 4 If the condition of lemma 2 are satisfied, then we have

$$\|n\|_{L_2\times L_\infty}\le E_2,\quad \|\varepsilon_x\|^2_{L_2\times L_\infty}\le E_3 \tag{2.7}$$

where the constants E_2 and E_3 depend on the norms $\|n_0\|_{L_2}$ and $\|\varepsilon_0\|_{H^1}$.

Proof In fact, by lemma 2

$$\int |\varepsilon_x|^2 dx + a\int n\,|\varepsilon|^2 dx = E(0),$$

hence we have

$$\|\varepsilon_x\|^2_{L_2}\le\frac{|a|}{2}\|n\|^2_{L_2}+\frac{|a|}{2}\|\varepsilon\|^4_{L_4}+|E(0)|\le\frac{|a|}{2}\|n\|^2_{L_2}+\frac{|a|}{2}\|\varepsilon\|^2_{L_\infty}\|\varepsilon\|^2_{L_2}+|E(0)|$$

$$\le\frac{|a|}{2}\|n\|^2_{L_2}+\frac{|a|}{2}(c^2\|\varepsilon\|^2_{L_2}+\delta^2\|\varepsilon_x\|^2_{L_2})\|\varepsilon\|^2_{L_2}+|E(0)|$$

$$\le\frac{|a|}{2}\|n\|^2_{L_2}+\frac{|a|\delta^2}{2}\|\varepsilon_0\|^2_{L_2}\|\varepsilon_x\|^2_{L_2}+\frac{|a|c^2}{2}\|\varepsilon_0\|^4_{L_2}+|E(0)|. \tag{2.7$'$}$$

By Lemma 3 $\int n^2 dx+\frac{\beta}{2a}\int i(\varepsilon\bar\varepsilon_x-\bar\varepsilon\varepsilon_x)dx=E_1(0)$, hence

$$\frac{|a|}{2}\|n\|^2_{L_2}\le\frac{|a|}{2}|E_1(0)|+\frac{|\beta|}{4}\int|\varepsilon|\,|\varepsilon_x|dx\le\frac{|a|}{2}|E_1(0)|+\frac{1}{8}(\beta^2\|\varepsilon\|^2_{L_2}+\|\varepsilon_x\|^2_{L_2})$$

Substituting the above inequality into (2.7)$'$, we abtain

$$\|\varepsilon_x\|^2_{L_2}\le\frac{|a|}{2}|E_1(0)|+\frac{1}{8}(\beta^2\|\varepsilon_0\|^2_{L_2}+\|\varepsilon_x\|^2_{L_2})+\frac{|a|}{2}\delta^2\|\varepsilon_0\|^2_{L_2}\|\varepsilon_x\|^2_{L_2}+\frac{|a|c^2}{2}\|\varepsilon_0\|^4_{L_2}$$
$$+|E(0)|.$$

Choosing δ suitably small, satisfying $\frac{|a|}{2}\|\varepsilon_0\|^2_{L_2}\delta^2<\frac{1}{2}$, then we have

$$\|\varepsilon_x\|^2_{L_2}\le\frac{8}{3}\left[\frac{|a|}{2}|E_1(0)|+\frac{1}{8}(\beta^2\|\varepsilon_0\|^2_{L_2}+\frac{|a|}{2}c^2\|\varepsilon_0\|^2_{L_2}+|E(0)|\right]=E_2\ .$$

From (2.6), it follows.

Corollary 1 $\|\varepsilon\|_{L_\infty}\le E_3'$.

where E_3' is a definite constant.

Lemma 5 If the conditions of Lemma 4 are satisfied, and assume that $\varepsilon_0(x)\in H^2$, then we have

$$\|n_t\|^2_{L_2\times L_\infty}\le E_4,\qquad \|\varepsilon_t\|^2_{L_2\times L_\infty}\le E_5\ . \tag{2.8}$$

where E_4 and E_5 are definite constants.

Proof By the equation $n_t+\beta|\varepsilon|^2_x=0$, it follows

$$\|n_t\|^2_{L_2}\le 4\beta^2\|\varepsilon_x\varepsilon\|^2_{L_2}\le 4\beta^2\|\varepsilon\|^2_{L_\infty}\|\varepsilon_x\|^2_{L_2}\le 4\beta^2E_3'E_3=E_4$$

Differentiating (2.1) with respect to t, then multiplying the resulting relation by $\bar\varepsilon_t$, taking the inner product and setting $E=\varepsilon_t$, we obtain

$$(iE_t+E_{xx}-an_t\varepsilon-anE.E)=0\ . \tag{2.9}$$

Taking the imaginary part of (2.9), we have

$$\frac{d}{dt}\|E\|^2_{L_2}\le|a|\int|n_t\varepsilon E|dx\le|a|\,\|\varepsilon\|_{L_\infty}\cdot\frac{1}{2}\left[\|n_t\|^2_{L_2}+\|E\|^2_{L_2}\right]$$

$$\leq |\alpha| E_3^1 \cdot \frac{1}{2}[\|n_t\|_{L_2}^2 + \|E\|_{L_2}^2] \leq c_1 + c_2\|E\|_{L_2}^2 .$$

By Gronwall's inequality and the conditions of this lemma, it follows (2.8).

Corollary 2 $\|\varepsilon_{xx}\|_{L_2 \times L_\infty} \leq E_6$,

where E_6 is a definite constant.

Proof From (2.1) we have $\|\varepsilon_{xx}\|_{L_2} \leq \|\varepsilon_t\|_{L_2} + |\alpha| \|\varepsilon\|_\infty \|n\|_{L_\infty} \leq E_6$.

Corollary 3 $\|\varepsilon_x\|_{L_\infty} \leq E_6$,

where E_6 is a definite conston.

Proof By corollary 3 and the Sobolev inequality, we obtain it imm diately.

Lemma 6 If the conditions of lemma 5 are satisfied, and assume that $n_0(x) \in H^1$, then we have

$$\|n_x\|_{L_2 \times L_\infty}^2 \leq E_7, \tag{2.10}$$

where E_7 is a definite constant.

Proof Differentiating (2.2) with respect to x, and taking inner product with n_x, it follows

$$(n_{tx} + \beta |\varepsilon|_{xx}^2, n_x) = 0 . \tag{2.11}$$

Since $|\varepsilon|_{xx}^2 = \varepsilon\bar{\varepsilon}_{xx} + 2|\varepsilon_x|^2 + \bar{\varepsilon}\varepsilon_{xx}$, then from (2.11) it follows

$$\frac{1}{2}\frac{d}{dt}\|n_x\|_{L_2}^2 \leq \|\varepsilon\|_{L_\infty}|\beta|(\|\varepsilon_{xx}\|_{L_2}^2 + \|n_x\|_{L_2}^2) + \|\varepsilon_x\|_{L_\infty}|\beta|(\|\varepsilon_x\|_{L_2}^2 + \|n_x\|_{L_2}^2) \leq C\|n_x\|_{L_2}^2 + c_2 .$$

By Gronwall's inequality and the conditions of this lemma, it follows (2.10).

Lemma 7 If the conditions of lemma 6 are satisfied, and assume that $\varepsilon_0(x) \in H^3$, then we have

$$\|n_{tx}\|_{L_2 \times L_\infty}^2 + \|\varepsilon_{xt}\|_{L_2 \times L_\infty}^2 \leq E_7 , \tag{2.12}$$

where E_7 is a definite constant.

Proof Differentiating (2.2) with respect to x, it follows

$$n_{tx} + \beta(\varepsilon_{xx}\bar{\varepsilon} + 2|\varepsilon_x|^2 + \varepsilon\bar{\varepsilon}_{xx}) = 0$$

Since

$$\|n_{tx}\|_{L_2}^2 \leq 3|\beta|[2\|\varepsilon\|_{L_\infty}^2\|\varepsilon_{xx}\|_{L_2}^2 + 4\||\varepsilon_x|^2\|_{L_2}^2] \leq 6|\beta|[\|\varepsilon\|_{L_\infty}^2\|\varepsilon_{xx}\|_{L_2}^2 + 2\|\varepsilon_x\|_{L_\infty}^2\|\varepsilon_x\|_{L_2}^2] \leq$$
$$6|\beta|[E_3' E_6 + 2E_6' E_3) = \text{const} ,$$

Differentiating (2.3) with respect to x and t, and taking inner product with $\bar{\varepsilon}_{tx}$, setting $\varepsilon_{tx} = \Sigma$, we obtain

$$(i\Sigma_t + \Sigma_{xx} - \alpha(n_{tx}\varepsilon + n_x\varepsilon_t + \varepsilon_x n_t + n\varepsilon_{tx}), \Sigma) = 0 .$$

Taking the imaginary part of the above equality, it follows

$$\frac{d}{dt}\|\Sigma\|_{L_2}^2 \leq \frac{|\alpha|}{2}\|\varepsilon\|_{L_\infty}(\|n_{xt}\|_{L_2}^2 + \|\Sigma\|_{L_2}^2) + |\alpha|\|\varepsilon_t\|_{L_\infty}\|n_x\|_{L_2}\|\Sigma\|_{L_2} +$$
$$\|\varepsilon_x\|_{L_\infty}\frac{|\alpha|}{2}(\|n_t\|_{L_2}^2 + \|\Sigma\|_{L_2}^2) \leq C_1(\|n_{xt}\|_{L_2}^2 + \|\Sigma\|_{L_2}^2) + \|n_x\|_{L_2}\cdot(\delta\|\varepsilon_{tx}\|_{L_2} + c\|\varepsilon_t\|_{L_2}) + c_2 \leq$$
$$c_3(\|n_{xt}\|_{L_2}^2 + \|\Sigma\|_{L_2}^2) + c_4 \leq c_5 + c_6\|\Sigma\|_{L_2}^2 .$$

By Gronwall's inequality, we obtain the boundness of $\|\Sigma\|_{L_2}$, this the lemma is pr ed.

Lemma 8 If the conditions of Lemma 7 are satisfied and assume that $\varepsilon_0(x) \in H^4$, then we have

$$\| n_{tt} \|^2_{L_2 \times L_\infty} + \| \varepsilon_{tt} \|^2_{L_2 \times L_\infty} \leq E_8 . \tag{2.13}$$

where E_8 is a definite constant.

Proof Differentiating (2.1) with respect to t twice, and differentiating (2.2) with respect to t, it follows

$$i\varepsilon_{ttt} + \varepsilon_{ttxx} - a(n_{tt}\varepsilon + 2n_t\varepsilon_t + n\varepsilon_{tt}) = 0 \tag{2.14}$$

$$n_{tt} + \beta(\varepsilon_{xt}\bar{\varepsilon} + \varepsilon_x\bar{\varepsilon}_t + \bar{\varepsilon}_x\varepsilon_t + \varepsilon\bar{\varepsilon}_{xt}) = 0 . \tag{2.15}$$

Setting $n_{tt} = N$, $\varepsilon_{tt} = E$, multiplying (2.14) by $\overline{E}$, and taking inner product, we obtain

$$(iE_t + E_{xx} - a(N\varepsilon + 2n_t\varepsilon_t + nE), E) = 0. \tag{2.16}$$

Taking the imaginary part of (2.16), it follows

$$\frac{1}{2}\frac{d}{dt}\| E \|^2_{L_2} \leq |a| \| \varepsilon \|_{L_\infty} \cdot \frac{1}{2}(\| N \|^2_{L_2} + \| E \|^2_{L_2}) - 2\| \varepsilon_t \|_{L_\infty} \cdot \frac{1}{2}(\| n_t \|^2_{L_2} + \| \varepsilon_t \|^2_{L_2}) \leq$$

$$c_1(\| E \|^2_{L_2} + \| N \|^2_{L_2}) + c_2 . \tag{2.17}$$

From (2.15), it yields

$$\| N \|^2_{L_2} = \beta^2 \| \varepsilon_{xt}\bar{\varepsilon} + \varepsilon_x\bar{\varepsilon}_t + \varepsilon_t\bar{\varepsilon}_x + \varepsilon\bar{\varepsilon}_{xt} \|^2_{L_2} \leq 8\beta^2 [\| \varepsilon_{xt}\varepsilon \|^2_{L_2} + \| \varepsilon_t\varepsilon_x \|^2_{L_2}]$$

$$\leq 8\beta^2 [\| \varepsilon \|^2_{L_\infty} \| \varepsilon_{xt} \|^2_{L_2} + \| \varepsilon_t \|^2_{L_\infty} \| \varepsilon_x \|^2_{L_2}] \leq c_3 .$$

Thus from (2.17) and Gronwall's inequality, we obtain the boundness of $\| \varepsilon_{tt} \|^2_{L_2}$, it follows (2.13).

Lemma 9 If the conditions of lemma 7 are satisfied, and assume that $\varepsilon_0(x) \in H^{2k}$ $(k \geq 2)$, then we have

$$\| D_t^{k-1} D_x n \|^2_{L_2} + \| D_t^{k-1} D_x \varepsilon \|^2_{L_2} + \| D_t^k n \|^2_{L_2} + \| D_t^k \varepsilon \|^2_{L_2} \leq E'_8 . \tag{2.18}$$

where E'_8 is a definite constant.

Proof As $k = 2$, (2.18) is true. Now we suppose (2.18) is also true as $k-1$. Differentiating (2.1) with respecs to t k times, differentiating (2.1) with respect to t $k-1$ times, and setting $D_t^k n = N$, $D_t^k \varepsilon = \Sigma$, we have

$$i\Sigma_t + \Sigma_{xx} - aD_t^k(n\varepsilon) = 0, \tag{2.19}$$

$$N + \beta D_t^{k-1} D_x |\varepsilon|^2 = 0 . \tag{2.20}$$

Since

$$D_t^k(n\varepsilon) = \varepsilon D_t^k n + \sum_{l=1}^{k-1} c_k^l D_t^l n \cdot D_t^{k-l}\varepsilon + nD_t^k\varepsilon,$$

and by the Sobolev inequality, $\| D_t^{k-2} n \|_{L_\infty}$ and $\| D_t^{k-2} n \|_{L_\infty}$ are bounded, we can obtain

$$|D_t^k(n\varepsilon)| \leq c(|D_t^k n| + |D_t^{k-1} n| + |D_t^k \varepsilon| + |D_t^{k-1}\varepsilon| + 1).$$

Multiplying (2.19) by $\overline{\Sigma}$, and taking the inner product and the imaginary part, we have

$$\frac{1}{2}\frac{d}{dt}\| \Sigma \|^2_{L_2} \leq c(\| N \|^2_{L_2} + \| \Sigma \|^2_{L_2} + 1) . \tag{2.21}$$

Differentiating (2.1) with respect to t $k-1$ times and x one time, setting $D_t^{k-1} D_x \varepsilon = M$, it follows

$$iM_t+M_{xx}-D_t^{k-1}D_x(n\varepsilon)=0. \tag{2.22}$$

Multiplying (2.22) by $\overline{M}$, taking the inner product and the imaginary part, we have

$$\frac{1}{2}\frac{d}{dt}\|M\|_{L_2}^2\le c(\|D_t^{k-1}D_xn\|_{L_2}^2+\|D_t^{k-1}D_x\varepsilon\|_{L_2}^2+1)\le c(\beta^2\|D_t^{k-2}D_x^2|\varepsilon|^2\|_{L_2}^2+\|M\|_{L_2}^2+1)$$
$$\le c(\|D_t^{k-1}\varepsilon\|_{L_2}^2+\|M\|_{L_2}^2+1)\le(1+\|M\|_{L_2}^2).$$

By Gronwall's inequality it yields

$$\|M\|_{L_2\times L_\infty}^2\le\text{const}. \tag{2.23}$$

From (2.20), it follows

$$\|N\|_{L_2}^2\le\|D_t^{k-1}D_x|\varepsilon|^2\|_{L_2}^2\le c(\|D_t^{k-1}D_x\varepsilon\|_{L_2}^2+\|D_t^{k-1}\varepsilon D_x\varepsilon\|_{L_2}^2+1)\le\text{const} \tag{2.24}$$

Instituting (2.23) into (2.21) and being Gronwall's inequality, we have

$$\|\Sigma\|_{L_2\times L_\infty}^2\le\text{const}. \tag{2.25}$$

Differentiating (2.2) with respect to t $k-2$ times and x one time, we have

$$\|D_t^{k-1}D_xn\|_{L_2}^2\le\beta^2\|D_t^{k-2}D_x^2|\varepsilon|^2\|_{L_2}^2\le c(\|D_t^k\varepsilon\|_{L_2}^2+1)\le\text{const}. \tag{2.26}$$

From (2.23), (2.24), (2.25) and (2.26), it follows (2.18).

3. The Existence of the Problem (2.1)(2.2)(2.3)

We first consider the equations (2.1)(2.2) with the periodic initial conditions

$$\begin{cases}\varepsilon|_{t=0}=\varepsilon_0(x),\quad n|_{t=0}=n_0(x),\quad -\infty<x<\infty,\\ \varepsilon(x+2D,t)=\varepsilon(\varepsilon,t),\ n(x+2D,t)=n(x,t),\ D>0,\ x\in R,\ t\ge0.\end{cases} \tag{3.1}$$

We apply the approximate solution

$$\varepsilon^l(x,t)=\sum_{j=1}^{l}a_j^l(t)\,w_j(x),\quad n^l(x,t)=\sum_{j=1}^{l}b_j^l(t)\,w_j(x) \tag{3.2}$$

to approach the solution $\varepsilon(x,t)$, $n(x,t)$ of the problem (2.1)(2.2)(2.3), where $\{w_j(x)\}$ is a basis, which is a system of the orthogonal eigenfunctions with periob 2D satisfying $-w_j''(x)=\lambda_jw_j$, and the unknown functions $\varepsilon^l(x,t)$, $n^l(x,t)$ satisfy the following integral relations:

$$\begin{cases}(i\varepsilon_t^l-\varepsilon_{xx}-un\varepsilon^l,w_j)=0,\\ (n_t+\beta|\varepsilon^l|_x^2,w_j)=0,\quad(j=1,2,\cdots,l)\\ (\varepsilon^l(0,x),w_j)=(\varepsilon_0(x),w_j),\quad(n^l(0,x),w_j)=(n_0(x),w_j).\end{cases} \tag{3.3}$$

It is easy to see that the system (3.3) is an initial value problem of nonlinear odinary system of the equations for the unknown coefficients $a_j^l(t)$, $b_j^l(t)$ by the priori integral estimates which are analogous to §2, we can obtain the uniform boundnesses of $\|D_t^k\varepsilon^l\|_{L_2}$, $\|D_t^kn^l\|_{L_2}$, $\|D_t^{k-1}D_xn^l\|_{L_2}$, $\|D_t^{k-1}D_x\varepsilon^l\|_{L_2}$ $(k=0,1,2,\cdots)$, by using analogous method in [6], it is easy to get the local solution $\varepsilon(x,t)\in L^\infty(0,t_0,H^s)$, $n(x,t)\in L^\infty(0,t_0,H^s)$ of the problem (2.1)(2.2)(3.1) on the interval $[0,t_0]$, where the time inteval length depends on the norms $\|\varepsilon_0\|_{H^k}$, and $\|n_0\|_{H^k}$. Due to the priori estimates in §2, we can obtain the following existence theorem of the global solution of the problem (2.1)(2.2)(3.1).

Theorem 1 If $\varepsilon_0(x)\in H^k(-D,D)$, $n_0(x)\in H^k(-D,D)$ $(k\ge2)$, then there exists the global solution $\varepsilon(x,t), n(x,t)$ of the problem (2.1)(2.2)(3.1),

$$\varepsilon(x,t) \in L^{\infty}(0,T,H^{k}),\ n(x,t) \in L^{\infty}(0,T,H^{k}).$$

Theorem 2 The smooth solution of the problem (2.1)(2.2)(3.1) is unique.

Proof In fact, suppose that there are two solutions, $\varepsilon_1(x,t)$, $n_1(x,t)$ and $\varepsilon_2(x,t)$, $n_2(x,t)$. Let

$$\varepsilon(x,t)=\varepsilon_1(x,t)-\varepsilon_2(x,t),\ n(x,t)=n_1(x,t)-n_2(x,t).$$

Thus from (2.1)(2.2), we have

$$\begin{cases} i\varepsilon_t+\varepsilon_{xx}-a(n_1\varepsilon_1-n_2\varepsilon_2)=0 \\ n_t+\beta(|\varepsilon_1|^2-|\varepsilon_2|^2)_x=0, \\ \varepsilon|_{t=0}=0,\ n|_{t=0}=0. \end{cases} \tag{3.4}$$

Since

$$n_1\varepsilon_1-n_2\varepsilon_2=n\varepsilon_1-n_2\varepsilon,\quad (|\varepsilon_1|^2-|\varepsilon_2|^2)_x= \text{[illegible]}.$$

multiplying the first, second equality of (3.4) by $\bar{\varepsilon}$, n respectively, and taking the inner product, it follows

$$(i\varepsilon_t+\varepsilon_{xx}-an\varepsilon_1-an_2\varepsilon_1\varepsilon)=0,\quad (n_t+\beta(\varepsilon_x\bar{\varepsilon}_1+\varepsilon_{xx}\bar{\varepsilon}+\varepsilon\bar{\varepsilon}_{1x}+\varepsilon_2\bar{\varepsilon}_x),n)=0. \tag{3.5}$$

Taking the imaginary part on the first equality of (3.4), we have

$$\frac{d}{dt}\|\varepsilon\|^2_{L_2}\le|(an\varepsilon_1,\varepsilon)|\le|a|\|\varepsilon_1\|_{L_\infty}\cdot\frac{1}{2}(\|n\|^2_{L_2}+\|\varepsilon\|^2_{L_2})\le c_1[\|n\|^2_{L_2}+\|\varepsilon\|^2_{L_2}]. \tag{3.6}$$

From the second equality of (3.5), it yields

$$\begin{aligned}\frac{d}{dt}\|n\|^2_{L_2}\le&|\beta|[\|\varepsilon_1\|_{L_\infty}\cdot\frac{1}{2}(\|\varepsilon_x\|^2_{L_2}+\|n\|^2_{L_2})+\|\varepsilon_x\|_{L_\infty}\cdot\frac{1}{2}(\|\varepsilon\|^2_{L_2}+\|n\|^2_{L_2}) \\ &+\|\varepsilon_{1x}\|_{L_\infty}\cdot\frac{1}{2}(\|\varepsilon\|^2_{L_2}+\|n\|^2_{L_2})+\|\varepsilon_2\|_{L_\infty}\cdot\frac{1}{2}(\|\varepsilon_x\|^2_{L_2}+\|n\|^2_{L_2}) \\ &\le c_2[\|\varepsilon\|^2_{L_2}+\|n\|^2_{L_2}+\|\varepsilon_x\|^2_{L_2}]\ .\end{aligned} \tag{3.7}$$

Differentiating the first equality of (3.4) with respect to x, then taking t the inner product with ε, we have

$$(\varepsilon_{xt}-i\varepsilon_{xxx}+ia(n_x\varepsilon_1+n\varepsilon_{1x}+n_{2x}\varepsilon+n_2\varepsilon_x,\varepsilon_x)=0\ .$$

Taking the real part of the above equality, it yields

$$\begin{aligned}\frac{d}{dt}\|\varepsilon_x\|^2_{L_2}\le&|a|\|\varepsilon_1\|_{L_\infty}\cdot\frac{1}{2}(\|n_x\|^2_{L_2}+\|\varepsilon_x\|^2_{L_2})+|a|\|\varepsilon_{1x}\|_{L_\infty}\cdot\frac{1}{2}(\|n\|^2_{L_2}+\|\varepsilon_x\|^2_{L_2}) \\ &+|a|\|n_{2x}\|_{L_\infty}\cdot\frac{1}{2}(\|\varepsilon\|^2_{L_2}+\|\varepsilon_x\|^2_{L_2})\le c_3[\|\varepsilon\|^2_{L_2}+\|\varepsilon_x\|^2_{L_2}+\|n_x\|^2_{L_2}+\|n\|^2_{L_2}]+c_4.\end{aligned} \tag{3.8}$$

Differentiating the second equality of (3.4) with respect to x, it follows

$$n_{xt}+\beta[\varepsilon_{xx}\bar{\varepsilon}_1+\varepsilon_x\bar{\varepsilon}_{1x}+\varepsilon_{2xx}\bar{\varepsilon}+\varepsilon_{2x}\bar{\varepsilon}_x+\varepsilon_x\bar{\varepsilon}_{1x}+\varepsilon\bar{\varepsilon}_{1xx}+\varepsilon_{2x}\bar{\varepsilon}_x+\varepsilon_2\bar{\varepsilon}_{xx}]=0\ .$$

Multiplying the above equalition by n_x and taking the inner product, it follows

$$(n_{xt}+\beta(\varepsilon_{xx}\varepsilon_1+\varepsilon_x\varepsilon_{1x}+\varepsilon_{2xx}\varepsilon+\varepsilon_{2x}\varepsilon_x+\varepsilon\varepsilon_{1xx}+\varepsilon_{2x}\varepsilon_x+\varepsilon_2\varepsilon_{xx}),n_x)=0\ .$$

Since

$$\|\varepsilon_{xx}\|_{L_2}\le\|\varepsilon_t\|_{L_2}+|a|\|\varepsilon_1\|_{L_\infty}\|n\|_{L_2}+|a|\|n_2\|_{L_\infty}\|\varepsilon\|_{L_2}\le c_3[\|\varepsilon_t\|_{L_2}+\|n\|_{L_2}+\|\varepsilon\|_{L_2}),$$

then we get

$$\frac{d}{dt}\|n_x\|_{L_2}^2 \le c_4[\|\varepsilon_x\|_{L_2}^2+\|n_x\|_{L_2}^2+\|n\|_{L_2}^2+\|\varepsilon\|_{L_2}^2+\|\varepsilon_t\|_{L_2}^2]. \quad (3.9)$$

Differentiating the first equality with respect to t, it follows

$$i\varepsilon_{tt}+\varepsilon_{txx}-an_t\varepsilon_1-an\varepsilon_{1t}-an_{2t}\varepsilon-an_2\varepsilon_t=0.$$

Taking the inner product with $\bar{\varepsilon}_t$ on the above equality, and then taking the imaginary part, we get

$$\frac{d}{dt}\|\varepsilon_t\|_{L_2}^2 \le c_5[\|n_t\|_{L_2}^2+\|\varepsilon_t\|_{L_2}^2+\|n\|_{L_2}^2+\|\varepsilon\|_{L_2}^2] \le c_5[\beta^2\|(|\varepsilon_1|^2-|\varepsilon_2|^2)_x\|_{L_2}^2+\|\varepsilon_t\|_{L_2}^2$$
$$+\|n\|_{L_2}^2+\|\varepsilon\|_{L_2}^2] \le c_6[\|\varepsilon_x\|_{L_2}^2+\|\varepsilon_t\|_{L_2}^2+\|n\|_{L_2}^2+\|\varepsilon\|_{L_2}^2]. \quad (3.10)$$

From (3.5)—(3.10), we have

$$\frac{d}{dt}[\|\varepsilon\|_{L_2}^2+\|\varepsilon_x\|_{L_2}^2+\|\varepsilon_t\|_{L_2}^2+\|n\|_{L_2}^2+\|n_x\|_{L_2}^2] \le c_7[\|\varepsilon\|_{L_2}^2+\|\varepsilon_x\|_{L_2}^2+\|\varepsilon_t\|_{L_2}^2+\|n\|_{L_2}^2+\|n_x\|_{L_2}^2].$$

By Gronwall's inequality and zero initial conditions, we have

$$\varepsilon(x,t)\equiv n(x,t)\equiv 0$$

Now we consider the initial value problom (2.1)(2.2)(2.3). By using the uniform estimates of the solution of the problem (2.1)(2.2)(3.1) for D, as in [7], we can prove that the solutions of the problem (2.1)(2.2)(2.3) as $D\to\infty$. Thus we have the following theorem.

Theorem 3 If the conditions of Theorem 2 are satisfied, then there exists a unique global solution for the initial value problem (2.1)—(2.3).

References

[1] V.D. Djordjevic and Redekopp, J.Fluid.Mech., 1(1): 703(1977).
[2] H.J.Grimshaw,Studies in appl.Math., 56: 241(1977).
[3] D.R.Nicholson and M.V.Grldman. Phys.Fliuds, 19.No.10, 1976.
[4] D.J.Denney,Studies in appl.Math., 56: 81(1977).
[5] Guo Boling (郭柏灵), SCIENTIA SINICA, (series A) Vol.xxv,No.9, 1982, 897—910.
[6] 郭柏灵，沈隆钧，应用数学学报，第五卷，第3期，1982，311—324.
[7] Zhou Yulin (周毓麟) and Guo Boling (郭柏灵). Acta Math.Sinica, Vol.27(1984).No.2, 154—176.