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The Mixed Bitsadze-Lavrent'ev-Tricomi Boundary Value Problem*

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Abstract. It is well known that F. G. Tricomi (1923) is the originator of the theory of boundary value problems for mixed type equations by establishing the Thicomi equation: $y \cdot u_{xx} + u_{yy} = 0$ which is hyperbolic for $y<0$, elliptic for $y>0$, and parabolic for $y=0$ and then applied it in the theory of transonic flows.

Then A. V. Bitsadze together with M. A. Lavrent'ev (1950) established the Bitsadze Lavrent'ev equation: $\mathrm{sgn}(y) \cdot u_{xx} + u_{yy} = 0$ where $\mathrm{sgn}(y) = 1$ for $y>0$, $= -1$ for $y<0$, $= 0$ for $y=0$ with the discontinuous coefficient $\mathrm{sgn}(y)$ of u_{xx}, while in the case of Tricomi equation the corresponding coefficient y is continuous. In this paper we establish the mixed Bitsadze Lavrent'ev Tricomi equation

$Lu = K(y) \cdot u_{xx} + \mathrm{sgn}(x) \cdot u_{yy} + r(x,y) \cdot u = f(x,y)$, where the coefficient $K = K(y)$ of u_{xx} is increasing continuous and coefficient $M = \mathrm{sgn}(x)$ of u_{yy} discontinuous, $r = r(x,y)$ is once continuously differentiable, $f = f(x,y)$ continuous.

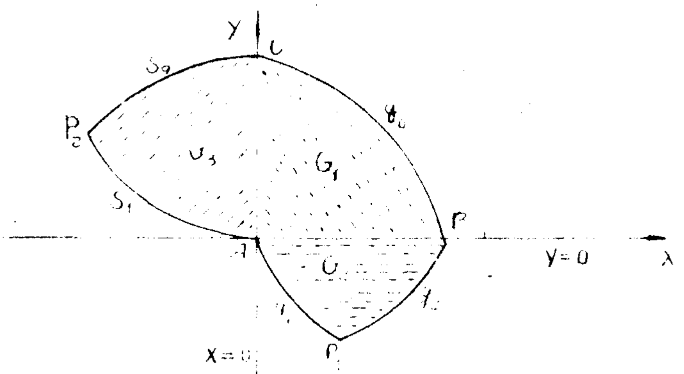
Finally we prove the uniqueness of quasi-regular solutions and observe that these new results can bbe applied in fluid dynamics.

The Mixed Bitsadze-Lavrent'ev-Tricomi Problem

Consider equation

$$(1) \qquad Lu = K(y) \cdot u_{xx} + \mathrm{sgn}(x) \cdot u_{yy} + r(x,y) \cdot u = f(x,y)$$

in a bounded simply connected region $G \subset R^2$ by the curves: A piecewise smooth curve g_0 lying in the region $G_1: x>0,\ y>0$ and intersecting the line $y=0$ at the point $B(1,0)$, and the line $x=0$ at the point $C(0,1)$, a smooth curve g_2 through B meeting a characteristic g_1 of the equation (1) issued from



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$A(0,0)$ at the point P_1 in the region $G_2: x>0,\ y<0$, the characteristic curve g_1, a smooth curve s_2 through C meeting a characteristic s_1 of the equation (1) issued from $A(0,0)$ at the point P_2 in the regien $G_3: x<0,\ y>0$, and the characteristic curve s_1 ([1], [3]).

It is clear that we can consider equations

(c_1) g_1: $\int_0^y \sqrt{-K(t)\cdot dt} = -x$ in G_2,

(c_2) g_2: $k\int_0^y \sqrt{-K(t)\cdot dt} = x-1$ $(k\geq 1)$ in G_2,

(c_3) s_1 : $\int_0^y \sqrt{K(t)\cdot dt} = -x$ in G_3,

(c_4) s_2 : $\int_0^y \sqrt{K(t)\cdot dt} = h\cdot x+1$ $(h\geq 1)$ in G_3,

such that (c_1) and (c_3) satisfy the characteristic equation

(2) $$K(y)\cdot(dy)^2+\mathrm{sgn}(x)\cdot(dx)^2=0$$

of (1), while (c_2) and (c_4) satisfy the expression

(2)′ $$K(y)\cdot(dy)^2+\mathrm{sgn}(x)\cdot(dx)^2\geq 0.$$

We assume the following conditions:

(3) $K=K(y)$ is an increasing continuous function in $\overline{G}$(= closure of G) and $M=\mathrm{sgn}(x)$ is discontinuous in $\overline{G}$ such that $M=1$ for $x>0$, $=-1$ for $x<0$, $=0$ for $x=0$, and $K\cdot M<0$ if $x\cdot y<0$, $K\cdot M>0$ if $x\cdot y>0$,

$$(*)\begin{cases} G_1:\ K(y)>0, y>0; & \mathrm{sgn}(x)=1>0 \ :\ \text{elliptic} \\ G_2:\ K(y)<0, y<0; & \mathrm{sgn}(x)=1>0 \ :\ \text{hyperbolic} \\ G_3:\ K(y)>0, y>0; & \mathrm{sgn}(x)=-1<0 \ :\ \text{hyperbolic} \\ AB:\ K(0)=0 ; & \mathrm{sgn}(x)=1>0 \ :\ \text{parabolic} \\ AC:\ K(y)>0 ; & \mathrm{sgn}(0)=0 \ :\ \text{parabolic} \\ A:\ K(0)=\mathrm{sgn}(0)=0 & \end{cases}$$

(4) $$r=r(x,y)\in C^1(\overline{G}),\ f=f(x,y)\in C^0(\overline{G}),$$

(5) $$r\leq 0 \text{ on } g_1\cup s_1,\ 2\cdot r+x\cdot r_x+y\cdot r_y\leq 0 \text{ in } \overline{G}_1,$$

$r+x\cdot r_x\leq 0$ in $\overline{G}_2$, $r+y\cdot r_y\leq 0$ in $\overline{G}_3$, $x\cdot dy-y\cdot dx\geq 0$ on g_0 "starlikedeness on g_0", where $\overline{G}_i$ (= closure of G_i) $(i=1,2,3)$.

In addition, we assume boundary condition

(6) $$u=0 \text{ on } g_0\cup g_2\cup s_2.$$

The Mixed Bitsadze-Lavrent′ev-Tricomi Problem or Problem (M):

consists in finding a solution $u=u(x,y)$ of (1) satisfying (6).

Theorem. Assume domain $G\subset R^2$ described above. If we assume conditions (3—6), then Problem (M) has at most one quasi-regular solution u.

Proof. Suppose u_1 and u_2 are two solutions of (1) satisfying boundary condition (6) (i.e: $u_i=0$ on $g_0\cup g_2\cup s_2$, $i=1,2$).

Denote $u=u_1-u_2$. Claim that

$$(7) \qquad u=0 \ (\text{or } u_1=u_2) \text{ in } G.$$

To prove (7) we apply the classical energy integral method in each region G_1, G_3 separately (because Green's theorem can not be applied in the whole region $G_1\cup G_3$ because of the discontinuity of the coefficient $M=\text{sgn}(x)$ in $G_1\cup G_3$), and in $G_1\cup G_2$ (because of the continuity of the coefficients $K=K(y)$ and $M=1$ in $G_1\cup G_2$), and then the maximum principle for elliptic and hyperbolic equations.

We note the integral expressions

$$(8) \qquad 2\cdot(lu, Mu)_{G_j}=2\iint_{G_j} lu\cdot Lu\cdot dxdy \qquad (j=1,3),$$

$$(8)' \qquad 2\cdot(lu, Mu)_{G_1\cup G_2}=2\cdot\iint_{G_1\cup G_2} lu\cdot Lu\cdot dxdy,$$

where

$$(9) \qquad lu=\begin{cases} x\cdot u_x+y\cdot u_y & \text{in } \overline{G}_1, \\ x\cdot u_x & \text{in } \overline{G}_2, \\ y\cdot u_y & \text{in } \overline{G}_3. \end{cases}$$

It is easy to see that ([2])

$$(10) \qquad 2\cdot(lu, Lu)_{G_1\cup G_2}=-\iint_{G_1}(2r+x\cdot r_x+y\cdot r_y)u^2\cdot dxdy$$
$$-\iint_{G_2}(r+x\cdot r_x)u^2\cdot dxdy+\int_{g_0} N_1^2\,(xdy-ydx)\cdot(Kv_1^2+v_2^2)ds$$
$$+2\int_{CA} y\cdot K\cdot u_x u_y v_1\cdot ds+\int_{G_1} y\cdot K'(y)\cdot u_x^2\cdot dxdy$$
$$+\iint_{G_2}(-K\cdot u_x^2+u_y^2)\cdot dxdy+\int_{g_1}[x\cdot K\cdot u_x^2 v_1+2x\cdot u_x u_y\cdot v_2-x\cdot u_y^2\, v_1]ds$$
$$+\int_{g_2} N_2^2(x\cdot v_1)(K\cdot v_1^2+v_2^2)ds+\int_{g_1} r\cdot(x\cdot v_1)\cdot u^2\cdot ds=\sum_{n=1}^{9} J_n^{(12)},$$

where N_i $(i=1,2)$ are normalizing factors such that

$$(11) \qquad u_x=N_1\cdot v_1,\ u_y=N_1\cdot v_2 \text{ on } g_0,$$

$$(11)' \qquad u_x=N_2\cdot v_1,\ u_y=N_2\cdot v_2 \text{ on } g_2,$$

(which are possible due to (6))

and $v=(v_1, v_2)$ in the outer normal vector on the boundary of $G_1\cup G_2$. Similarly

$$(12) \quad 2(lu, Lu)_{G_3}=-\iint_{G_3}(r+y\cdot r_y)\cdot u^2\cdot dxdy+\iint_{G_3}[\,(y\cdot K)'\,u_x^2+u_y^2\,]dxdy$$
$$+2\cdot\int_{AC} yK\cdot u_x u_y\cdot v_1\cdot ds+\int_{s_2} N_3^2\cdot(y\cdot v_2)(K\cdot v_1^2-v_2^2)ds$$
$$+\int_{s_1}[2yK\cdot u_x u_y\cdot v_1-y\cdot K\cdot u_x^2\cdot v_2-y\cdot u_y^2\cdot v_2]ds+\int_{s_1} r(y\cdot v_2)u^2\cdot ds=\sum_{m=1}^{6} J_m^{(3)},$$

where N_3 is a normalizing factor such that $u_x=N_3\cdot v_1$, $u_y=N_3\cdot v_2$ on s_2, (which is possible due to (6)) and $v=(v_1, v_2)$ is the outer normal vector on the boundary of G_3.

It is clear now that

$J_1^{(12)} \geq 0$ because: $2 \cdot r + x \cdot r_x + y \cdot r_y \leq 0$ in $\overline{G}_1$ by (5),

$J_2^{(12)} \geq 0$ because: $r + x \cdot r_x \leq 0$ in $\overline{G}_2$ by (5),

$J_3^{(12)} \geq 0$ because: $xdy - ydx \geq 0$ on g_0 by (5) and $K = K(y) > 0$ on g_0 by (3),

$J_4^{(12)} + J_3^{(3)} = 0$ because: v_1(on AC) $= -v_1$(on CA),

$J_5^{(12)} \geq 0$ because: $K'(y) > 0$ in G_1, and $K = K(y)$ is assumed increasing in G_1 by (3),

$J_6^{(12)} \geq 0$ because $K = K(y) < 0$ in G_2 by (3),

$J_7^{(12)} \geq 0$ because $x > 0$ on g_1 and $K(y) < 0$ on g_1 by (3), $v_1 < 0$ on g_1 (by the geometry of g_1), and $\begin{vmatrix} K \cdot v_1 & v_2 \\ v_2 & -v_1 \end{vmatrix} = -(Kv_1^2 + v_2^2) = 0$ by (2),

$J_8^{(12)} \geq 0$ because: $x > 0$ on g_2 by (3), $v_1 > 0$ on g_2 (by the geometry of g_2), and $K \cdot v_1^2 + v_2^2 \geq 0$ by $(2)'$,

$J_9^{(12)} \geq 0$ because: $r \leq 0$ on g_1 by (5), $x > 0$ on g_1 by (3), and $v_1 < 0$ on g_1 (by the genmetry of g_1),

$J_1^{(3)} \geq 0$ because: $r + y \cdot r_y \leq 0$ in G_3 by (5),

$J_2^{(3)} \geq 0$ because: $(y \cdot K)' = K - y \cdot K' \geq 0$ in G_3. K is increasing in G_3 by (3), and $K > 0$ in G_3 by (3),

$J_4^{(3)} \geq 0$ because: $y > 0$ on s_2 by (3), $v_2 > 0$ on s_2 (by the geometry of s_2), $K \cdot v_1^2 - v_2^2 \geq 0$ by $(2)'$,

$J_5^{(3)} \geq 0$ because: $y > 0$ on s_1 by (3), and $K(y)) > 0$ on s_1 by (3), $v_2 < 0$ on s_1 (by the geometry of s_1), and $\begin{vmatrix} -K \cdot v_2 & K \cdot v_1 \\ K \cdot v_1 & -v_2 \end{vmatrix} = -K(Kv_1^2 - v_2^2) = 0$

by (2), and

$J_6^{(3)} \geq 0$ because: $r \leq 0$ on s_1 by (5), $y > 0$ on s_1 by (3), and $v_2 < 0$ on s_1 (by the geometry of s_1).

Therefore, by adding (10) and (12) and by taking into account the above results on the integrals

$J_n^{(12)}$ $(n = 1, 2, \cdots, 9)$, $J_m^{(3)}$ $(m = 1, 2, \cdots, 6)$ we conclude that $u = 0$ on the boundary of G. Thus by employing the maximum principle we prove (7).

Refernces

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