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Lacunary Interpolation by Splines (II)*

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Let Δ: $0=x_0<x_1<\cdots<x_n=1$ be a subdivision of $[0,1]$, $T=\{0,1,2,3\}$, z_{1i}, $z_{2i}\in T$, $z_{1i}<z_{2i}$ and

$S_\Delta=\{s(x)\mid s(x)\in C^3[0,1];\ s(x)\in\pi_5,\ x\in[x_i,\ x_{i+1}],\ i=0,1,\cdots,n-1\}$. For $s(x)\in S_\Delta$, denote the interpolation conditions

$$s^{(z_{1i})}(x_i)=f^{(z_{1i})}(x_i),\quad s^{(z_{2i})}(x_i)=f^{(z_{2i})}(x_i),\quad i=0,1,\cdots,n$$

by $\begin{pmatrix} z_{20}z_{21}\cdots z_{2n} \\ z_{10}z_{11}\cdots z_{1n}\end{pmatrix}$ and denote two additional interpolation conditions $s^{(z')}(x_i)=f^{(z')}(x_i)$, $s^{(z'')}(x_j)=f^{(z'')}(x_j)$ by $b(x_i,\ z';x_j,\ z'')$, where $z'\in T\setminus\{z_{1i},\ z_{2i}\}$, $z''\in T\setminus\{z_{1j},\ z_{2j}\}$. Now, we call the following interpolation problems

$$\begin{pmatrix} z_{20}z_{21}\cdots z_{2n} \\ z_{10}z_{11}\cdots z_{1n}\end{pmatrix}+b(x_0,\ z';x_n,\ z''),\quad \begin{pmatrix} z_{20}z_{21}\cdots z_{2n} \\ z_{10}z_{11}\cdots z_{1n}\end{pmatrix}+b(x_i,z';x_j,z''),\quad 0\leqslant i<j\leqslant n$$

(but two equalities do not hold simultaneously), and

$$\begin{pmatrix} z_{20}z_{21}\cdots z_{2n} \\ z_{10}z_{11}\cdots z_{1n}\end{pmatrix}+b(x_i,\ z';x_i,\ z'')$$

the type Ⅰ, type Ⅱ, type Ⅲ respectively.

Recently, the authers[1] considered the existence and uniqueness of the interpoletion problems of these three types. For fixed $k\in\mathbf{N}$, let $W=\begin{pmatrix} z_{20}z_{21}\cdots z_{2n} \\ z_{10}z_{11}\cdots z_{1n}\end{pmatrix}$. In this paper we consider the convergent problems of the recurrent interplation. By a recurrent interpolation we mean the interpolation:

$$Z=(\overbrace{W,W,\cdots,W}^{n},\begin{matrix} z_{20} \\ z_{10}\end{matrix})=\begin{pmatrix} z_{20}\cdots z_{2,k-1}z_{20}\cdots z_{2,k-1}\cdots z_{20}\cdots z_{2,k-1}z_{20} \\ z_{10}\cdots z_{1,k-1}z_{10}\cdots z_{1,k-1}\cdots z_{10}\cdots z_{1,k-1}z_{10}\end{pmatrix}$$

with two additional interpolation conditions such that the interpolation to be regular of type Ⅰ, type Ⅱ or type Ⅲ, and

$$\Delta:\ 0=x_0<x_1<\cdots<x_k<x_{k+1}<\cdots<x_{kn}=1$$

to be equidistant, *i.e.*, $x=ih$, $h=\frac{1}{kn}$.

Let $$B_{\overline{W}}=B_{\begin{pmatrix} z_{20}z_{21} \\ z_{10}z_{11}\end{pmatrix}}B_{\begin{pmatrix} z_{21}z_{22} \\ z_{11}z_{12}\end{pmatrix}}\cdots B_{\begin{pmatrix} z_{2\,k-2}z_{2\,k-1} \\ z_{1\,k-2}z_{1\,k-1}\end{pmatrix}}B_{\begin{pmatrix} z_{2\,k-1}\,z_{20} \\ z_{1\,k-1}\,z_{10}\end{pmatrix}}, \tag{1}$$

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where $B_{\binom{z_2z_4}{z_1z_3}}$ is so colled T-matrix[1].

It is easy to verify that

$$|\det B_{\binom{z_2z_4}{z_1z_3}}|=|\mathrm{dgt}\, B_{\binom{z_21}{z_10}}+\det B_{\binom{z_41}{z_30}}|, \text{ thus } |\det B_{\binom{z_2z_4}{z_1z_3}}\cdot\det B_{\binom{z_4z_6}{z_3z_5}}|=|\det B_{\binom{z_2z_6}{z_1z_5}}|.$$

Because of $|\det B_{\binom{z_2z_2}{z_1z_1}}|=1$ for all z_1, $z_2\in T$, consequently $|\det B_{\overline{W}}|=1$. Acordingly we have

Lemma I The eigenvalues λ_1, λ_2 of $B_{\overline{W}}$ satisfy $\lambda_1\lambda_2=\pm 1$.

Suppose that $\lambda_1 \neq \lambda_2$ are the eigenvalues of $B_{\overline{W}}$, then there exists nonsingular matrix $P=\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22}\end{pmatrix}$, $\det P=1$ such that

$$B_{\overline{W}}=P\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}P^{-1} \quad . \tag{2}$$

Furthermore, if any element of P is zero then the corresponding element of P' which keeps (2) validity would be zero also.

In the following theorems, we assume that $W=\begin{pmatrix} z_{20}z_{21}\cdots z_{2\ k-1} \\ z_{10}z_{11}\cdots z_{1\ k-1}\end{pmatrix}$

$$Z=(\overbrace{W,W,\cdots W}^{n},\ \begin{matrix} z_{20} \\ z_{10}\end{matrix}) \tag{3}$$

and $s^{(r)}(x_i)=\frac{1}{2}(s^{(r)}_{(x_i+)}+s^{(r)}_{(x_i-)})$, $r=4,5$, $i=1,2,\cdots,kn-1$

Theorem I Let $f(x)\in C^6[0,1]$, Z be defined in (3) and $\{m_1, m_2\}=T\backslash\{z_{10}, z_{20}\}$, $m_1<m_2$. Suppose the following conditions hold:

(i) the eigenvalues of $B_{\overline{W}}$ satisfy $|\lambda_1|>1>|\lambda_2|$;

(ii) $s(x)\in S_\Delta$ is determined by the regular interpolation conditions of type I: $z+b(x_0, z'; x_{kn}, z'')$;

(iii) $p_{12}p_{21}\neq 0$, when $z'=m_1$, $z''=m_2$; $p_{11}p_{21}\neq 0$, when $z'=m_2$, $z''=m_1$.

Then there exist constants C_r depending on $\overline{W}$ only such that

$$\|s^{(r)}(x)-f^{(r)}(x)\|_\infty < C_r\|f^{(6)}\|_\infty h^{6-r},\quad r=0,1,\cdots,5.$$

Theorem 2 Let $f(x)\in C^6$, Z be defined in (3) and $s(x)\in S_\Delta$ be determined by the regular interpolation conditions

$$\tilde{z}=z+b(x_i, z'; x_j, z'') \tag{4}$$

of type I, type II or type III. If $\lambda_1=1$, $\lambda_2=-1$, then there exist constants C'_r depending on $\overline{W}$ only such that

$$\|s^{(r)}(x)-f^{(r)}(x)\|_\infty < C'_r\|f^{(6)}\|_\infty h^{5-r},\quad r=0,1,\cdots,5 .$$

Theorem 3 Let $f(x)\in C^6$, and $s(x)\in S_\Delta$ be determined by the regular interpolation conditions (4) of type I, type II or type III. If $B_{\overline{W}}\neq\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ or

$\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\lambda_1=\lambda_2=1$, then

$$\| s^{(r)}(x)-f^{(r)}(x)\|_\infty \leqslant C_r'' \| f^{(6)}\|_\infty h^{4-r}, \qquad r=0,1,\cdots,4.$$

If $B_{\overline{W}}=\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ or $\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, then

$$\| s^{(r)}(x)-f^{(r)}(x)\|_\infty \leqslant C_r''' \| f^{(6)}\|_\infty h^{5-r}, \qquad r=0,1,\cdots,5.$$

Some examples.

1. $W=\begin{pmatrix}2\\ 0\end{pmatrix}$. This is the case that[2] considered. This time

$$B_{\overline{W}}=B_{\begin{pmatrix}2 & 2\\ 0 & 0\end{pmatrix}}=\begin{pmatrix}-\frac{3}{2} & 30\\ -\frac{1}{24} & \frac{3}{2}\end{pmatrix}$$

and its eigenvalues $\lambda_1=1, \lambda_2=-1$. Because of $B_{\overline{W}}^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, we have $B_{\overline{W}}^n=B_{\overline{W}}$ for odd n and $B_{\overline{W}}^n=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ for even n. Therefore, for odd n and the recurrent interpolation of type Ⅰ, the boundary interpolation conditions may be arbitrarily selected. For even n, the boundary interpolation condditions may be selected to be $b(x_0, 1; x_{kn}, 3)$ or $b(x_0, 3; x_{kn}, 1)$. For all these cases as well as for the recurent interpolation of type Ⅲ, the degree of approximation is $O(h^5)$ and can not be improved.

2. $W=\begin{pmatrix}3\\ 0\end{pmatrix}$. Then $B_{\overline{W}}=B_{\begin{pmatrix}3 & 3\\ 0 & 0\end{pmatrix}}=\begin{pmatrix}-\frac{7}{3} & -\frac{20}{3}\\ -\frac{2}{3} & -\frac{7}{3}\end{pmatrix}$,

its eigenvalues $\lambda_{1,2}=\frac{-7\pm 2\sqrt{10}}{3}$ and any element of $B_{\overline{W}}$ does not vanish. So the degree of approximation by the recurrent interpolation of type Ⅰ attains $O(h^6)$, no matter what boundary interpolation conditions are chosen. Similarly, if $W=\begin{pmatrix}1\\ 0\end{pmatrix}$ then the degree of approximation by the recurrent interpolation of type Ⅰ also attains $O(h^6)$.

3. $W=\begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix}$. Then $B_{\overline{W}}=B_{\begin{pmatrix}1 & 3\\ 0 & 2\end{pmatrix}}B_{\begin{pmatrix}3 & 1\\ 2 & 0\end{pmatrix}}=\begin{pmatrix}13 & 84\\ 2 & 13\end{pmatrix}$, $\lambda_{1,2}=13\pm\sqrt{168}$,

and so the degree of approximations by any recurrent interpoaltion of type Ⅰ attains $O(h^6)$.

Before proving the theorems mentioned above, we give some lemmas.

Lemma 2 Let $$Q_u(x;t)=\begin{cases}-(x-t)^5, & t<x<u;\\ (x-t)^5, & u<t<x;\\ 0, & \text{otherwise.}\end{cases}$$

Suppose that $f(x)\in C^6[0,1]$ and $S_{\tilde{z}}(x;f)\in S_\Delta$ be determined by the regular interpolation conditions (4), then

$$s_{\tilde{z}}^{(r)}(u;f)-f^{(r)}(u)=\frac{1}{5!}\int_0^1 f^{(6)}(t)s_{\tilde{z}}^{(r)}(u;Q_u(\cdot;t))\mathrm{d}t. \qquad (5)$$

Proof For $u\in[0,1]$ we have

$$f(x)=\sum_{i=0}^{5}\frac{1}{i!}f^{(i)}(u)(x-u)^i+\frac{1}{5!}\int_0^1 f^{(6)}(t)Q_u(x;t)\mathrm{d}t\ .$$

According to the regularity of $\tilde{Z}$, linear functional $S_{\tilde{z}}(u;f)-f(u)$ vanishes as $f\in\pi_5$. Noting that $Q_u(u;t)=0$, we have

$$s_{\tilde{z}}(u;f)-f(u)=\frac{1}{5!}\int_0^1 f^{(6)}(t)s_{\tilde{z}}(u;Q_u(\cdot;t))\mathrm{d}t\ .$$

Thus we obtain (5) as $r<4$. When $r=4,5$, we can process in subintervals and obtain (5) similarly. Lemma 2 established.

Now we denote the set of quintic lacunary interpolation splines with all of knots locate at integers lie in $[0,n]$(or $[0,kn]$ according to the number of knots) and denote the set of all this kind of splines by $\bar{S}_n$. We have

Lemma 3 Let $f(x)\in C^6[0,1]$ and $s_{\tilde{z}}(x;f)\in S_\Delta$ be determined by the regular interpolation conditions (4), then

$$\|s_{\tilde{z}}^{(r)}(u;f)-f^{(r)}(u)\|_\infty\leqslant\frac{h^{6-r}}{5}\|f^{(6)}\|_\infty\max_{u\in[0,n]}\int_0^n|\bar{s}_{\tilde{z}}^{(r)}(u;Q_u(\cdot;t))|\mathrm{d}t$$

Proof Put $u=vh$, from Lemma 2 we have

$$s_{\tilde{z}}(vh;f)-f(vh)=\frac{h^6}{5!}\int_0^n f^{(6)}(th)\bar{s}_{\tilde{z}}(u,Q_v(\cdot;t))\mathrm{d}t$$

thus

$$\|s_{\tilde{z}}^{(r)}(\cdot;f)-f^{(r)}(\cdot)\|_\infty\leqslant\frac{h^{6-r}}{5!}\|f^{(6)}\|_\infty\max_{v\in[0,n]}\int_0^n|\bar{s}_{\tilde{z}}^{(r)}(v;Q_v(\cdot;t))|\mathrm{d}t$$

and lemma 3 esablished.

Set cardinal functions of the regular interpolation conditions

$$\tilde{z}=z+b(i',z';i'',z'') \qquad (6)$$

to be $L_{\eta i}(x)\in\bar{S}_n$, $(\eta=1,2,\ i=0,1,\cdots,n+1)$ satisfying

$$L_{\eta i}(x)|_{b(i',z';i'',z'')}=0,\quad L_{\eta i}^{(z_{\xi j})}(j)=0,\qquad(\eta\neq\xi),$$
$$L_{\eta i}^{(z_{\eta j})}(j)=\delta_{ij},\qquad \eta=1,2;\ i,j=0,1,\cdots,n,$$
$$L_{1,n+1}(x)|_z=0,\ L_{1,n+1}^{(z')}(i')=1,\ L_{1,n+1}^{(z'')}(i'')=0,$$
$$L_{2,n+1}(x)|_z=0,\ L_{2,n+1}^{(z')}(i')=0,\ L_{2,n+1}^{(z'')}(i'')=1,$$

where
$$Z=\begin{pmatrix} z_{20} & z_{21} & \cdots & z_{2n}\\ z_{10} & z_{11} & \cdots & z_{1n}\end{pmatrix}$$

Lemma 4 Suppose that cardinal functions fo the regular interpolation conditions (6) for $x\in[j,j+1]$, $\eta=1,2$, $i,j=0,1,\cdots,n$. satisfy

$$|L_{\eta i}^{(r)}(x)|<C_6\lambda^{|i-j|},\quad |L_{1,n+1}^{(r)}(x)|<C_6\lambda^{|i'-j|},\quad |L_{2,n+1}^{(r)}(x)|<C_6\lambda^{|i''-j|}\quad (7)$$

where $0<\lambda<1$, C_6 is a constant independing of n then there exists constant C_7, independent of n, such that

$$\int_0^h|\overline{s}_{\bar z}^{(r)}(u;Q_u(\cdot;t))|\mathrm{d}t<C_7 \qquad (8)$$

holds for all $u\in[0,n]$.

Proof We abbreviate $\overline{s}_{\bar z}$ as $\overline{s}$. From the definition of $L_{\eta i}(x)$, we have

$$\overline{s}^{(r)}(u;Q_u(\cdot;t))=\sum_{i=0}^{n}\sum_{\eta=1}^{2}\mathrm{D}_i^{z_{\eta i}}Q_u(x;t)L_{\eta i}^{(r)}(u)+\mathrm{D}_{i'}^{z'}Q_u(x;t)L_{1,n+1}(u)$$
$$+\mathrm{D}_{i''}^{z''}Q_u(x;t)L_{2,n+1}^{(r)}(u),$$

where

$$\mathrm{D}_j^{z_{\eta i}}Q_u(x;t)=\frac{\partial^{z_{\eta i}}}{\partial x^{z_{\eta i}}}Q_u(x;t)\Big|_{x=j}\quad .$$

Suppose that $u\in[n_0,n_0+1]$, as $t<u$, $t\in[j,j+1]$. For definiteness, assume $i'<u<i''$, from (7), we have

$$\int_j^{j-1}|\overline{s}^{(r)}(u,Q_u(\cdot;t))|dt\leqslant\sum_{\eta=1}^{2}\sum_{i=0}^{j}60|L_{\eta i}^{(r)}(u)|\cdot\int_j^{j+1}(t-i)^5\mathrm{d}t+$$

$$+60|L_{1,n+1}^{(r)}(u)|\int_j^{j+1}(t-i)^5\mathrm{d}t<40C_6\lambda^{n_0-j}\sum_{i=0}^{\infty}i^6\lambda^i=C_6C_8\lambda^{n_0-j}\quad . \qquad (9)$$

Similarly, as $t>u$, $t\in[j,j+1]$, we have

$$\int_j^{j+1}|\overline{s}^{(r)}(u;Q_u(\cdot;t))|\mathrm{d}t<C_6C_8\lambda^{j_1-n_0}\quad . \qquad (10)$$

When t,u lie in the same interval $[n_0,n_0+1]$ we can discuss in the intervaals $[n_0,u)$ and $[u,n_0+1]$ respectively and obtain the same results as (9) and (10), thus

$$\int_0^n|\overline{s}^{(r)}(u;Q_u(\cdot;t))|\mathrm{d}t<2C_6C_8(1+\lambda+\lambda^2+\cdots)<C_7,\qquad u\in[0,n].$$

Lemma 4 established.

Remark For fixed $k\in\mathrm{N}$, we consider the situation that all of knots locate at integers lie in $[0,kn]$. If

$$|L_{\eta i}^{(r)}(x)|<\lambda^{|n_1-n_0}| \qquad (11)$$

holds for $x\in[kn_0,k(n_0+1)]$, $i=kn_1+r_1$, $0\leqslant r_1<k$, we can prove similarly that (8) still holds for C_7 depending on k .

Lemma 5 Suppose that there exists constant C_9 such that

$$|L_{\eta i}^{(r)}(x)|<C_9,\ \eta=1,2,\ i=0,1,\cdots,n+1,\ x\in[0,n],$$

then there exists constant C_{10}, independent of n, such that

$$\int_0^n \left|\overline{s}_{\bar{z}}^{(r)}(u; Q_u(\cdot;t))\right| dt < C_{10} n .$$

Proof We consider the situation $r=0$ at first. If $t \in [j, j+1]$, $j+1 < u$ we have

$$\overline{s}(u; Q_u(\cdot;t)) = \overline{s}(u; (t-x)_+^5)$$

Put $t = j + \tau_1$, $0 \leqslant \tau \leqslant 1$, then

$$(t-x)_+^5 = (j-x)_-^5 + 5(j-x)_-^x\tau + 10(j-x)_-^3\tau^2 + 10(j-x)^2\tau^3 + 5(j-x)_\perp\tau^4 + (j-x)_\perp^\circ\tau^5, \quad (13)$$

where

$$(\varphi(x))_- = \begin{cases} \varphi(x), & x \in [0, j+\tau] \\ 0, & \text{otherwise} \end{cases}$$

Because of $(j-x)_-^5$ and its derivatives equal to $(j-x)_+^5$ and its derivaties respectively at the integer knots, therefore, by the regularity of interpolation, we have

$$\overline{s}(u; (j-x)_-^5) = \overline{s}(u; (j-x)_+^5) = (j-u)_+^5 = 0. \quad (14)$$

Similarly

$$\overline{s}(u; (j-u)_-^4) = \overline{s}(u; (j-x)_+^4) = (j-u)_+^4 = 0 \quad (15)$$

and

$$\overline{s}(u; (j-x)_-^\nu) = \overline{s}(u; (j-x)_+^\nu), \quad \nu = 0, 1, 2, 3. \quad (16)$$

In the above equalities, we mean the derivatives of the right hand side at the discontinuous point to be left derivatives. Put

$$s_0(x) = \begin{cases} (j-x)^\circ & x \in [0, j-1], \\ \frac{3}{4}\mu^5 - \frac{5}{4}\mu^4 + 1, & x \in [j-1, j] \quad \mu = x-(j-1),\ 0 \leqslant \mu \leqslant 1, \\ -\frac{3}{4}(1-\mu)^5 + \frac{5}{4}(1-\mu)^4, & x \in [j, j+1] \quad \mu = x-j,\ 0 \leqslant \mu \leqslant 1, \\ 0 & \text{otherwise}; \end{cases}$$

$$s_1(x) = \begin{cases} (j-x), & x \in [0, j-1, \\ -\frac{1}{10}\mu^5 + \frac{1}{4}\mu^4 - \mu + 1, & x \in [j-1, j],\ \mu = x-(j-1), \quad 0 \leqslant \mu \leqslant 1, \\ -\frac{1}{10}(1-\mu)^5 + \frac{1}{4}(1-\mu)^4, & x \in [j, j+1],\ \mu = x-j,\ 0 \leqslant \mu \leqslant 1. \\ 0 & \text{otherwise}; \end{cases}$$

$$s_2(x) = \begin{cases} (j-x)^2, & x \in [0, j-1], \\ -\frac{5}{8}\mu^5 + \frac{1}{8}\mu^4 + \mu^2 - 2\mu + 1, & x \in [j-1, j],\ \mu = x-(j-1),\ 0 \leqslant \mu \leqslant 1, \\ \frac{1}{8}(1-\mu)^5 - \frac{1}{8}(1-\mu)^4, & x \in [j, j+1],\ \mu = x-j,\ 0 \leqslant \mu \leqslant 1, \\ 0, & \text{otherwise}; \end{cases}$$

$$s_3(x)=\begin{cases}(j-x)^3, & x\in[0,j-1],\\ \frac{1}{10}\mu^5-\frac{1}{8}\mu^4-\mu^3+3\mu^2-3\mu+1, & x\in[j-1,j]\quad \mu=x-(j-1),\ 0<\mu<1,\\ \frac{1}{10}(1-\mu)^5-\frac{1}{8}(1-\mu)^4, & x\in[j,j+1],\ \mu=x-j,\ 0<\mu<1,\\ 0, & \text{otherwise}.\end{cases}$$

Obviously, $s_\nu(x)\in\overline{S}_n$, $\nu=0,1,2,3$. According to the regularity of interpolatton and $\mu>j+1$, we obtaip

$$\overline{s}(u;s_\nu(x))=s_\nu(\mu)=0,\qquad \nu=0,1,2,3. \tag{17}$$

For $\nu=0,1,2,3$. at knots $s_\nu(x)$ and its derivatives equal to $(j-x)_+^\nu$ and its derivatives respectively but $x=j$. Thus from (16)(17) we have

$$\begin{aligned}\overline{s}(u;(j-x)_+^\nu)&=\overline{s}(u;(j-x)_+^\nu-s_\nu(x))\\ &=D_j^{z_{1j}}((j-x)_+^\nu-s_\nu(x))L_{1j}(u)+D_j^{z_{2j}}(j-x)_+^\nu-s_\nu(x))L_{2j}(u),\end{aligned} \tag{18}$$

when the addtional condition just locates in the knot $x=j$, the right hand side of (18) must add the corresponding term $L_{1,n+1}(u)$ or $L_{2,n+1}(u)$. From (12),(13). (14),(15),(18), when $t=j+\tau$, $0<\tau<1$, $j+1<u$ we have

$$\overline{s}(u;Q_u(\cdot;t))=10\tau^2\sum_{\eta=1}^{2}D_j^{z_{\eta j}}((j-x)_+^3-s_3(x))L_{\eta j}(u)+10\tau^3\sum_{\eta=1}^{2}D_j^{z_{\eta j}}((j-x)_+^2-s_2(x))L_{\eta j}(u)$$

$$+5\tau^4\sum_{\eta=1}^{2}D_j^{z_{\eta j}}((j-x)_+-s_1(x))L_{\eta j}(x)+\tau^5\sum_{\eta=1}^{2}D_j^{z_{\eta j}}((j-x)_+^0-s_0(x))L_{\eta j}(u),$$

perhaps with the linear combination of $L_{1,n+1}(u)$ or $L_{2,n+1}(u)$. Thus, from the hypotheses of Lemma 5 we obtain

$$\int_j^{j+1}|\overline{s}(u;Q_u(\cdot;t))|\,dt<C_9C_{11}, \tag{19}$$

where C_{11} is a constant. As $u<j<t\leqslant j+1$, $t\in[[u],u]$ and $t\in[u,[u]+1]$ (19) holds similarly. Therefore

$$\int_0^n|\overline{s}(u;Q_u(\cdot;t))|\,dt<C_{10}n.$$

Thus, as $r=0$, Lemma 5 established. In the case of $r>0$, we can prove similarly. We finish the proof of Lemma 5.

Lemma 6 Put $W=\begin{pmatrix}z_{20} & z_{21} & \cdots & z_{2,k-1} & z_{2k}\\ z_{10} & z_{11} & \cdots & z_{1,k-1} & z_{1k}\end{pmatrix}$, $z_{2k}=z_{20}$, $z_{1k}=z_{10}$, and

$$\{m_{1i},m_{2i}\}=T\backslash\{z_{1i},z_{2i}\},\ m_{1i}<m_{2i},\ i=0,1,\cdots,k. \tag{20}$$

Suppose $\overline{s}(x)\in\overline{S}_n$ satisfies $\overline{s}(x)|_{\overline{W}}=0$, $x\in[j,j+k]$. Set

$$\alpha_i=\overline{s}^{(m_{1i})}(j+i),\qquad \beta_i=\overline{s}^{(m_{2i})}(j+i),\qquad i=0,1,\cdots,k,$$

$$\gamma_i=\max\{\overline{s}^{(4)}(j+i+),\overline{s}^{(4)}(j+i-)\},$$

$$\delta_i=\max\{\overline{s}^{(5)}(j+i+),\overline{s}^{(5)}(j+i-)\},\qquad i=1,2,\cdots,k-1,$$

and γ_0, δ_0, γ_k, δ_k mean the oneside derivatives in the interval $[j,j+k]$. Then there exists a constant C_{12} depending on $\overline{W}$ only such that

$$\max_{i=0,1,\cdots,k}\{|a_i|,|\beta_i|,|\gamma_i|,|\delta_i|\}<C_{12}\max\{|a_0|,|\beta_0|\}\ , \tag{21}$$

$$\max_{\substack{x\in[j,j+k]\\ r=0,1,\cdots,5}}|s^{(r)}(x)|<C_{12}\max\{|a_0|,|\beta_0|\}\ . \tag{22}$$

Instead of a_0, β_0, by, a_k, β_k, (21),(22) still hold.

Proof According to the definition of T-matrix, we have

$$(a_i,\ \beta_i)=(a_0,\ \beta_0)B_{\binom{z_{20}\ z_{21}}{z_{10}\ z_{11}}}\cdots B_{\binom{z_{2,i-1},\ z_{2i}}{z_{1\ i-1},\ z_{1i}}}\quad .$$

Because of this kind of matrices have only in totality k and γ_i, δ_i are detemined by a_{i-1}, β_{i-1}, a_i,β_i, a_{i+1} and β_{i+1}, thus (21) holds. Besides, it is obviovs that a polynomial in an interval is uniquely detemined by its values and derivatives at the end of the interval. Thus (22) holds and Lemma 6 established.

Now we turn to prove theorems mentioned above.

Proof of Theorem I By Lemma 3, it is sufficient to prove that $\int_0^{kn}|\bar{s}_{\bar{z}}^{(r)}(u;Q_u(\cdot;t))|dt<C_{13}$, where C_{13} is independent of n. By Lemma 4 and the remark after it, we need only to verify that the inequalities (11) hold.

Set $\{m_{1i},\ m_{2i}\}=T\backslash\{z_{1i},\ z_{2i}\}$, $m_{1i}<m_{2i}$. Suppose that in the interval $[0,1]$. $P_0(t)\in\pi_5$ satisfies

$$P_0(t)\Big|_{\binom{z_{20}\ z_{21}}{z_{10}\ z_{11}}}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad \left(\text{or }\begin{pmatrix}0&0\\0&1\end{pmatrix}\right), \tag{23}$$

$$P_0^{(m_{10})}(0)=a_0',\qquad\qquad P_0^{(m_{20})}(0)=\beta_0\quad . \tag{23}$$

It is obvious that $a_1=P_0^{(m_{11})}(1)$, $\beta_1=P_0^{(m_{21})}(1)$ satisfy

$$(a,\beta_1)=(a_0,\ \beta_0)B_{\binom{z_{20}\ z_{21}}{z_{10}\ z_{11}}}+(a',\ b'). \tag{24}$$

Furthermore, in the interval $[1,2]$, $P_1(t)\in\pi_5$ is determined uniquely by the following interpolation conditions:

$$P_1(t)\Big|_{\binom{z_{21}\ z_{22}}{z_{11}\ z_{12}}}=\begin{pmatrix}1&0\\0&0\end{pmatrix}\qquad\left(\text{or }\begin{pmatrix}0&0\\1&0\end{pmatrix}\right).$$

$$P_1^{(m_{11})}(1)=a_1,\qquad\qquad P_1^{(m_{21})}(1)=\beta_1\quad . \tag{25}$$

Set $a_2=P^{(m_{12})}(2)$, $\beta_2=P_1^{(m_{22})}(2)$, noting (24) we obtain

$$(a_2,\beta_2)=(a_0,\beta_0)B_{\binom{z_{20}\ z_{21}}{z_{10}\ z_{11}}}B_{\binom{z_{21}\ z_{22}}{z_{11}\ z_{12}}}+(a,b)\quad . \tag{26}$$

Because the variation of all these 8 z_{ij} are finite, so all of a,b are bounded. On account of the polynomials $P_0(t)$, $P_1(t)$ determined by (23),(25) are unique, and so it is true in any interval $[i,\ i+2]$.

Now suppose that $i=kn_1+l$, $0\leqslant l<k$. Put

$$B_1 = B_{\binom{z_{20}\ z_{21}}{z_{10}\ z_{11}}} \cdots B_{\binom{z_{2\ l-2}\ z_{2\ l-1}}{z_{1\ l-2}\ z_{1\ l-1}}},\quad B_2 = B_{\binom{z_{2\ l+1}\ z_{2\ l+2}}{z_{1\ l-1}\ z_{1\ l+2}}} \cdots B_{\binom{z_{2\ k-1}\ z_{20}}{z_{1\ k-1}\ z_{10}}},$$

$a_j = L_{n_i}^{(m_{1j})}(j)$, $\beta_j = L_{n_i}^{(m_{2j})}(j)$ and $n_2 = n - n_1 - 1$. From the definition of T-matrix and (26) we have

$$(a_{i-1}, \beta_{i-1}) = (a_0, \beta_0) B_{\overline{W}} B_1,$$

$$(a_{i+1}, \beta_{i+1}) = (a_{i-1}, \beta_{i-1}) B_{\binom{z_{2\ l-1}\ z_{2l}}{z_{1\ l-1}\ z_{1l}}} B_{\binom{z_{2l}\ z_{2\ l+1}}{z_{1l}\ z_{1\ l+1}}} + (a, b),$$

$$(a_{kn}, \beta_{kn}) = (a_{i+1}, \beta_{i+1}) B_2 B_{\overline{W}}^{n_2}.$$

Noting that $B_1 B_{\binom{z_{2\ l-1}\ z_{2l}}{z_{1\ l-1}\ z_{1l}}} B_{\binom{z_{2l}\ z_{2\ l+1}}{z_{1l}\ z_{1\ l+1}}} B_2 = B_{\overline{W}}$, we have

$$(a_{kn}, \beta_{kn}) = (a_0, \beta_0) B_{\overline{W}}^n + (a_1, b_1) B_{\overline{W}}^{n_2}, \tag{27}$$

where $(a_1, b_1) = (a, b) B_2$. Because the variations of B_2 caused by the variations of l have only k different kinds, and so $\{(a, b)\}$ is a bounded set.

According to the hypothesis of Theorem 1 that the eigenvalues of $B_{\overline{W}}$ satisfying $|\lambda_1| > 1 > |\lambda_2|$, so there exists a nonsingular matrix $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$, det $P = 1$, such that

$$B_{\overline{W}} = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

This time (27) becomes

$$(a_{kn}, \beta_{kn}) = (a_0, \beta_0) P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} + (a_1, b_1) P \begin{pmatrix} \lambda_1^{n_2} & 0 \\ 0 & \lambda_2^{n_2} \end{pmatrix} P^{-1}, \tag{28}$$

i.e.

$$\begin{aligned} a_{kn} = {} & a_0(\lambda_1^n p_{11} p_{22} - \lambda_2^n p_{12} p_{21}) + \beta_0(\lambda_1^n - \lambda_2^n) p_{21} p_{22} + \\ & a_1(\lambda_1^{n_2} p_{11} p_{22} - \lambda_2^{n_2} p_{12} p_{21}) + b_1(\lambda_1^{n_2} - \lambda_2^{n_2}) p_{21} p_{22}, \end{aligned} \tag{29}$$

$$\begin{aligned} \beta_{kn} = {} & a_0(-\lambda_1^n + \lambda_2^n) p_{11} p_{12} + \beta_0(-\lambda_1^n p_{12} p_{21} + \lambda_2^n p_{11} p_{22}) + \\ & + a_1(-\lambda_1^{n_2} + \lambda_2^{n_2}) p_{11} p_{12} + b_1(-\lambda_1^{n_2} p_{12} p_{21} + \lambda_2^{n_2} p_{11} p_{22}). \end{aligned} \tag{30}$$

As the boundary interpolation conditions are $b(0, m_1; kn, m_1)$, we have $a_0 = a_{kn} = 0$. On account of the regularity of the interpolation problem, it is obvious that $(\lambda_1^n - \lambda_2^n) p_{21} p_{22} \neq 0$. Noting that $|\lambda_1 \lambda_2| = 1$, we can solve β_0 from (29). Substituting β_0 in (30) we obtain β_{kn}. Then we have

$$|\beta_0| < C_{14} |\lambda_2|^{n-n_2}, \quad |\beta_{kn}| < C_{14} |\lambda_2|^{n_2}, \tag{31}$$

where C_{14} is a constant independent of n.

As the boundary interpolation conditions are $b(0, m_1; kn, m_2)$ we have $a_0 = \beta_{kn} = 0$. On acconut of the regularity of the interpolation problem, it is obvious that $-\lambda_1^n p_{12} p_{21} + \lambda_2^n p_{11} p_{22} \neq 0$. By the hypothesis (iii) of Theorem 1, $p_{12} p_{21} \neq 0$, from (29), (30) we obtain

$$|\beta_0| < C_{14} |\lambda_2|^{n-n_2}, \quad |a_{kn}| < C_{14} |\lambda_2|^{n_2}$$

similarly. As for the other two types of boundary interpolation conditions and the situations for $L_{\eta 0}(x)$, $L_{\eta,kn}(x)$ anh $L_{\eta,kn+1}(x)$ can be discussed similarly. Therefore from $(a_{kj}, \beta_{kj})=(a_0, \beta_0)B_{W}^{j}$ we conclude that there exists constant C_{15} such that

$$|a_{kj}|<C_{15}|\lambda_2|^{n_1-j+1}, \quad |\beta_{kj}|<C_{15}|\lambda_2|^{n_1-j+1}, \quad kj<i \tag{33}$$

$$|a_{kj}|<C_{15}|\lambda_2|^{j-n_1-1}, \quad |\beta_{kj}|<C_{15}|\lambda_2|^{j-n_1-1}, \quad kj>i \ . \tag{34}$$

From Lemma 6 we have

$$|L_{\eta l}^{(r)}(x)|<C_{16}|\lambda_2|^{|n_1-j|}, \quad x\in[kj, k(j+1)]$$

Using the similar proof of Lemma 6, from (33),(34) we can deduce that $|L_{\eta i}^{(r)}(x)|<C_{16}$ holds for $x\in[kn_1, k(n_1+1)]$. Thus, we complete the proof for Theorem 1.

Proof of Theorem 2 On account of Lemma 5, it is sufficient to prove that the cardinal functions $L_{\eta l}(x)$ at integer knots determined by the recurrent interpolation $\tilde{z}$ satisfy the following inequalities

$$|L_{\eta i}^{(r)}(x)|<C_{17}.$$

As the interpolation condition is of type Ⅰ, because of $\lambda_1^n=1$, $\lambda_2^n=\pm 1$ the boundness of $L_{\eta i}^{(5)}(x)$ is deduced from (29) and (30) as well as the proof of Theorem 1. As the interpolation condition is of type Ⅲ, the boundness of $L_{\eta i}^{(5)}(x)$ follows from (28) immediately. As the interpolation condition is of type Ⅱ, the same conclusion holds just as the analysis we give above for type Ⅰ and type Ⅲ.

Proof of Theorem 3 is similar to that of Theorem 2 and 3.

References

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