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Constraining the First Hitting Time*

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Let $X(\omega)=\{x(t,\ \omega),\ t\geq 0\}$ be Markov chains with stationary, defined on complete probability space $(\Omega,\mathscr{F},P)$. The transition probabitity matrix $\{p_{ij}(t);\ t\geq 0,\ i,j\in \mathrm{I}\}$ is standard and satisfies the forward equations, where $\mathrm{I}=\{0,1,2,\cdots\}$ is the state space of $X(\omega)$. All states of $X(\omega)$ are stable. The sample functions are right lower semicontinuous. The Q-matrix is conservative. The $X(\omega)$ is Borel measurable and well separate. The condition (C) is true. (cf. [1])

Lemma I Let F be a compact state set, then

$$\lim_{m\to\infty}\{x(\frac{r}{2^m},\ \omega)\in \mathrm{F},\ \ [t_1 2^m]\leq r\leq [t_2 2^m],\ \ r \text{ is an integer}\}$$
$$\doteq\{x(r,\ \omega)\in \mathrm{F},\ \ t_1\leq r\leq t_2\}. \tag{1}$$

where $0\leq t_1<t_2\leq\infty$. If $t_2=\infty$, then $[t_2 2^m]=\infty$, and denote $t_1\leq r\leq t_2$ by $t_1\leq r<\infty$. If we write $A\doteq B$, then it means that their symmetric difference is a null set.

The proof is clear ▮

Theorem I Let F be a compact state set, $0\leq t_1<t_2\leq t$, then

$p_i\{\tau_{\mathrm{E}}(\omega)<t,\ \ x(\tau_{\mathrm{E}},\ \omega)=j,\ x(r,\ \ \omega)\in \mathrm{F},\ \ t_1\leq r\leq t_2\}$

$$=\sum_{l\in\tilde{\mathrm{E}}}\ \sum_{m\in \mathrm{F}}\ \sum_{k\in \mathrm{F}}\int_0^{t_1}{}_{\mathrm{E}}p_{il}^{(u)}\,q_{lj}\,p_{jm}^{(t_1-u)}\ {}_{\tilde{\mathrm{F}}}\,p_{mk}^{(t_2-t_1)}\mathrm{d}u$$
$$+\sum_{l\in\tilde{\mathrm{E}}}\ \sum_{m\in\tilde{\mathrm{E}}\cap \mathrm{F}}\ \sum_{k\in \mathrm{F}}\int_{t_1}^{t_2}{}_{\mathrm{E}}p_{il}^{(t_1)}\ {}_{\mathrm{E}\cap\tilde{\mathrm{F}}}p_{lm}^{(u-t_1)}\,q_{mj}\ {}_{\tilde{\mathrm{F}}}p_{jk}^{(t_2-u)}\mathrm{d}u$$
$$+\sum_{l\in\tilde{\mathrm{E}}}\ \sum_{m\in\tilde{\mathrm{E}}\cap \mathrm{F}}\ \sum_{k\in\tilde{\mathrm{E}}}\int_{t_2}^{t}{}_{\mathrm{E}}p_{il}^{(t_1)}\ {}_{\mathrm{E}\cup\tilde{\mathrm{F}}}p_{lm}^{(t_2-t_1)}\ {}_{\mathrm{E}}p_{mk}^{(u+t_1-t_2)}q_{kj}\mathrm{d}u\quad (i\in\tilde{\mathrm{E}},\ \ j\in \mathrm{E}\cap \mathrm{F}), \tag{2}$$
$$=\sum_{l\in\tilde{\mathrm{E}}}\ \sum_{m\in \mathrm{F}}\ \sum_{k\in \mathrm{F}}\int_0^{t_1}{}_{\mathrm{E}}p_{il}^{(u)}q_{lj}\,p_{jm}^{(t_1-u)}\ {}_{\tilde{\mathrm{F}}}p_{mk}^{(t_2-t_1)}\mathrm{d}u$$
$$+\sum_{l\in\tilde{\mathrm{E}}}\ \sum_{m\in\tilde{\mathrm{E}}\cap \mathrm{F}}\ \sum_{k\in\tilde{\mathrm{E}}}\int_{t_2}^{t}{}_{\mathrm{E}}p_{il}^{(t_1)}\ {}_{\mathrm{E}\cup\tilde{\mathrm{F}}}p_{lm}^{(t_2-t_1)}\ {}_{\mathrm{E}}p_{mk}^{(u+t_1-t_2)}q_{kj}\,\mathrm{d}u\quad (i\in\tilde{\mathrm{E}},\ \ j\in \mathrm{E}\cap\tilde{\mathrm{F}}), \tag{3}$$
$$=\delta_{ij}\sum_{k\in \mathrm{F}}\ \sum_{l\in \mathrm{F}}p_{ik}^{(t_1)}\ {}_{\tilde{\mathrm{F}}}p_{kl}^{(t_2-t_1)}\qquad (i\in \mathrm{E},\ \ j\in \mathrm{E}) \tag{4}$$
$$=o\ (i\in\tilde{\mathrm{E}},\ \ j\in\tilde{\mathrm{E}} \text{ or } i\in \mathrm{E},\ j\in\tilde{\mathrm{E}}) \tag{5}$$

Proof Since $p_i\{\tau_{\mathrm{E}}(s,\ \omega)<t,\ \ x(\tau_{\mathrm{E}}(s,\ \omega),\omega)=j,\ \ x(rs,\omega)\in \mathrm{F},\ \ [t_1 s^{-1}]\leq r\leq [t_2 s^{-1}]\}$

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$$= \sum_{a=1}^{[t_1 s^{-1}]-1} p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<a,\ x(as,\omega)\in\{j\}\cap E,\ x(rs,\omega)\in F,\ [t_1 s^{-1}]\le r$$

$$\le [t_2 s^{-1}]\} + \sum_{a=[t_1 s^{-1}]+1}^{[t_2 s^{-1}]-1} p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<[t_1 s^{-1}],\ x(\beta_1 s,\omega)\in\tilde{E}\cap F,$$

$$[t_1 s^{-1}]\le\beta_1<a,\ x(as,\omega)\in\{j\}\cap E\cap F,\ x(rs,\omega)\in F,\ a<r\le[t_2 s^{-1}]\}$$

$$+ \sum_{a=[t_2 s^{-1}]+2}^{[t\tilde{s}^{-1}]} p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<[t_1 s^{-1}],\ x(\beta_1 s,\omega)\in\tilde{E}\cap F,\ [t_1 s^{-1}]\le\beta_1\le$$

$$[t_2 s^{-1}],\ x(\beta_2 s,\omega)\in\tilde{E},\ [t_2 s^{-1}]<\beta_2<a,\ x(as,\omega)\in\{i\}\cap E\}$$

$$+ p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<[t_1 s^{-1}],\ x([t_1 s^{-1}]s,\omega)\in E\cap F\cap\{j\},\ x(rs,\omega)\in F,$$
$$[t_1 s^{-1}]<r\le[t_2 s^{-1}]\} + p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<[t_1 s^{-1}],\ x(\beta_1 s,\omega)\in\tilde{E}\cap F,$$
$$[t_1 s^{-1}]\le\beta_1<[t_2 s^{-1}],\ x([t_2 s^{-1}]s,\omega)\in E\cap F\cap\{j\}\} + p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta$$
$$<[t_1 s^{-1}],\ x(\beta_1 s,\omega)\in\tilde{E}\cap F,\ [t_1 s^{-1}]\le\beta_1\le[t_2 s^{-1}],\ x([t_2 s^{-1}]s+s,\omega)\in E$$
$$\cap\{j\}\}.$$

$$I_1(s)+I_2(s)+I_3(s)+I_4(s)+I_5(s)+I_6(s), \quad (6)$$

$$I_1(s) = \sum_{a=2}^{[t_1 s^{-1}]-1}\sum_{l\in\tilde{E}} p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<a-1,\ x((a-1)s,\omega)=l\}\ p_{lj}(s)$$

$$\sum_{m\in F} p_i\{x([t_1 s^{-1}]s-as,\omega)=m\}\ p_m\{x(rs,\omega)\in F,\ 1\le r\le[t^2 s^{-1}]-$$

$$-[t_1 s^{-1}]\} + p_i\{x(0,\omega)\in\tilde{E},\ x(s,\omega)\in\{j\}\cap E\}\sum_{k\in F} p_j\{x([t_1 s^{-1}]s-s,\omega)=$$

$$=k\},\ p_k\{x(rs,\omega)\in F,\ 1\le r\le[t_2 s^{-1}]-[t_1 s^{-1}]\} = \sum_{l\in\tilde{E}}\sum_{m\in F}\sum_{k\in F}\int_{2s}^{[t_1 s^{-1}]s}$$

$${}_E p_{il}^{([us^{-1}]-1)}(s)\frac{1}{s}p_{lj}^{(s)}p_{jm}([t_1 s^{-1}]s-[us^{-1}]s)\ {}_{\tilde{F}}p_{mk}^{([t_2 s^{-1}]-[t_1 s^{-1}])}(s)\,du + p_{ij}^{(s)}\cdot$$

$$\sum_{m\in F} p_{im}([t_1 s^{-1}]s-s]\sum_{k\in F}{}_{\tilde{F}}p_{mk}^{([t_2 s^{-1}]-[t_1 s^{-1}])}(s),$$

$$\lim_{s\to 0} I_1(s) = \sum_{l\in\tilde{E}}\sum_{m\in F}\sum_{k\in F}\int_0^{t_1}{}_E p_{il}^{(u)}\ q_{lj}\ p_{jm}^{(t_1-u)}\ {}_{\tilde{F}}p_{mk}^{(t_2-t_1)}du \quad (i\in\tilde{E},\ j\in E). \quad (7)$$

$$I_2(s) = \sum_{a=[t_1 s^{-1}]+1}^{[t_2 s^{-1}]-1}\sum_{l\in\tilde{E}} p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<[t_1 s^{-1}]-1,\ x\big([t_1 s^{-1}]s-s,\omega)$$

$$=l\}\sum_{m\in\tilde{E}\cap F} p_l\{x(\beta_1 s,\omega)\in\tilde{E}\cap F,\ 1\le\beta_1<a+1-[t_1 s^{-1}]-1,\ x(as-[t_1 s^{-1}]s,\omega)$$

$$=m\}\ p_{mj}(s)\sum_{k\in F} p_j\{x(rs,\omega)\in F,\ 1\le r<[t_2 s^{-1}]-a,\ x([t_2 s^{-1}]s-as,\omega)=k\}$$

$$= \sum_{l\in\tilde{E}} \sum_{m\in\tilde{E}\cap F} \sum_{k\in F} \int_{[t_1 s^{-1}]s+s}^{[t_2 s^{-1}]s} {}_E p_{il}^{[t_1 s^{-1}]}(s)_{E\cup\tilde{F}} p_{lm}^{([us^{-1}]-[t_1 s^{-1}])}(s)\frac{1}{s}p_{mj}(s)$$

$$_{\tilde{F}}p_{jk}^{([t_2 s^{-1}]-[us^{-1}])}(s)\,du,$$

$$\lim_{s\to 0} I_2(s) = \sum_{l\in\tilde{E}} \sum_{m\in\tilde{E}\cap F} \sum_{k\in F} \int_{t_1}^{t_2} {}_E p_{il}^{(t_1)}\,{}_{E\cup\tilde{F}}p_{lm}^{(u-t_1)} q_{mjF}\, p_{jk}^{(t_2-u)} du \quad (i\in\tilde{E},\ j\in E\cap F). \tag{8}$$

$$I_3(s) = \sum_{a=[t_2 s^{-1}]+2}^{[t\tilde{s}^{-1}]} \sum_{l\in\tilde{E}} p_i\{x(\beta s,\omega)\in\tilde{E},\ 0\le\beta<[t_1 s^{-1}]-1,\ x([t_1 s^{-1}]s-s,\omega)=l\}$$

$$\sum_{m\in\tilde{E}\cap F} p_l\{x(\beta_1 s,\omega)\in\tilde{E}\cap F,\ 1\le\beta_1\le[t_2 s^{-1}]-[t_1 s^{-1}],\ x([t_2 s^{-1}]s-[t_1 s^{-1}]s+s,\omega)=m\}\cdot$$

$$\cdot\sum_{k\in\tilde{E}} p_m\{x(\beta_2 s,\omega)\in\tilde{E},\ [t_1 s^{-1}]\le\beta_2\le a+[t_1 s^{-1}]-[t_2 s^{-1}]-2,\ x((a+[t_1 s^{-1}]$$

$$-[t_2 s^{-1}]-1)s,\omega)=k\}\cdot p_{kj}^{(s)}$$

$$= \sum_{l\in\tilde{E}} \sum_{m\in\tilde{E}\cap F} \sum_{l\in\tilde{E}} \int_{[t_2 s^{-1}]s+2s}^{[ts^{-1}]s+s} {}_E p_{il}^{([t_1 s^{-1}]-1)}(s)_{E\cup\tilde{F}} p_{lm}^{([t_2 s^{-1}]-[t_1 s^{-1}]+1)}(s)$$

$$_E p_{mk}^{([us^{-1}]+[t_1 s^{-1}]-[t_2 s^{-1}]-1)}(s)\frac{1}{s}p_{kj}^{(s)}\,du.$$

$$\lim_{s\to 0} I_3(s) = \sum_{l\in\tilde{E}} \sum_{m\in\tilde{E}\cap F} \sum_{k\in\tilde{E}} \int_{t_2}^{t} {}_E p_{il}^{(t_1)}\,{}_{E\cup\tilde{F}}p_{lm}^{(t_2-t_1)}\,{}_E p_{mk}^{(u+t_1-t_2)} q_{kj} du \quad (i\in\tilde{E},\ j\in E). \tag{9}$$

$$\lim_{s\to 0} I_4(s) = \lim_{s\to 0} I_5(s) = \lim_{s\to 0} I_6(s) = 0. \tag{10}$$

Hence theorem 1 is proved by (6) - (10).

Note: $[t\tilde{s}^{-1}] = ts^{-1}-1$, if ts^{-1} is an integer;

$= [ts^{-1}]$, if ts^{-1} is not an integer.

Corollary 1. For $0\le t_1<t<t_2$, $p_i\{\tau_E(\omega)<t,\ x(\tau_E,\omega)=j,\ x(r,\omega)\in F,\ t_1\le r\le t_2\}$

$$= \sum_{l\in\tilde{E}} \sum_{m\in F} \sum_{k\in F} \int_0^{t_1} {}_E p_{jl}^{(u)} q_{lj}\, p_{im}^{(t_1-u)}\,{}_{\tilde{F}}p_{mk}^{(t_2-t_1)} du$$

$$+ \sum_{l\in\tilde{E}} \sum_{m\in\tilde{E}\cap F} \sum_{k\in F} \int_{t_1}^{t} {}_E p_{il}^{(t_1)}\,{}_{E\cup\tilde{F}}p_{lm}^{(u-t_1)} q_{mj}\,{}_{\tilde{F}}p_{jk}^{(t_2-u)} du \quad (i\in\tilde{E},\ j\in E\cap F), \tag{11}$$

$$= \sum_{l\in\tilde{E}} \sum_{m\in F} \sum_{k\in F} \int_0^{t_1} {}_E p_{il}^{(u)} q_{lj}\, p_{jm}^{(t_1-u)}\,{}_{\tilde{F}}p_{mk}^{(t_2-t_1)} du \quad (i\in\tilde{E},\ j\in E\cap\tilde{F}), \tag{12}$$

$$= \delta_{ij} \sum_{k\in F} \sum_{l\in F} p_{ik}^{(t_1)}\,{}_{\tilde{E}}p_{kl}^{(t_2-t_1)} \quad (i,j\in E), \tag{13}$$

$= 0 \quad (i \in I,\ j \in \tilde{E})$.

Corollary 2 . For $0 < t \leq t_1$,

$$p_i\{\tau_E(\omega) < t,\ x(\tau_E, \omega) = j.\ x(r, \omega) \in F,\ t_1 \leq t \leq t_2\} \tag{14}$$

$$= \sum_{l \in \tilde{E}} \sum_{m \in F} \sum_{k \in F} \int_0^t {}_E p_{il}^{(u)} q_{lj} p_{jm}^{(t_1 - u)} {}_{\tilde{F}} p_{mk}^{(t_2 - t_1)} du \quad (i \in \tilde{E},\ j \in E), \tag{15}$$

$$= \delta_{ij} \sum_{k \in F} \sum_{l \in F} p_{ik}^{(t_1)} {}_{\tilde{F}} p_{kl}^{(t_2 - t_1)} \quad (i, j \in E), \tag{16}$$

$$= 0 \quad (i \in I,\ i \in \tilde{E}). \tag{17}$$

Corollary 3. $p_i\{\tau_E(\omega) < \infty,\ x(\tau_E, \omega) = j.\ x(r, \omega) \in F,\ t_1 \leq r \leq t_2\}$

$$= \sum_{l \in \tilde{E}} \sum_{m \in F} \sum_{k \in F} \int_0^{t_1} {}_E p_{il}^{(u)} q_{lj} p_{jm}^{(t_1 - u)} {}_{\tilde{F}} p_{mk}^{(t_2 - t_1)} du +$$

$$= \sum_{l \in \tilde{E}} \sum_{m \in \tilde{E} \cap F} \sum_{k \in F} \int_{t_1}^{t_2} {}_E p_{il}^{(t_1)} {}_{E \cup \tilde{F}} p_{jm}^{(u - t_1)} q_{mj} {}_{\tilde{F}} p_{jk}^{(t_2 - u)} du$$

$$+ \sum_{l \in \tilde{E}} \sum_{m \in \tilde{E} \cap F} \sum_{k \in \tilde{E}} \int_{t_2}^{\infty} {}_E p_{il}^{(t_1)} {}_{E \cup \tilde{F}} p_{lm}^{(t_2 - t_1)} {}_E p_{mk}^{(u - t_1 - t_2)} q_{kj} du \quad (i \in \tilde{E},\ j \in E \cap F) \tag{18}$$

$$= \sum_{l \in \tilde{E}} \sum_{m \in F} \sum_{k \in F} \int_0^{t_1} {}_E p_{il}^{(u)} q_{lj} p_{jm}^{(t_1 - u)} {}_{\tilde{F}} p_{mk}^{(t_2 - t_1)} du$$

$$+ \sum_{l \in \tilde{E}} \sum_{m \in \tilde{E} \cap F} \sum_{k \in \tilde{E}} \int_{t_2}^{\infty} {}_E p_{il}^{(t_1)} {}_{E \cup \tilde{F}} p_{lm}^{(t_2 - t_1)} {}_E p_{mk}^{(u + t_1 - t_2)} q_{kj} du \quad (i \in \tilde{E},\ j \in E \cap \tilde{F}) \tag{19}$$

$$= \delta_{ij} \sum_{k \in F} \sum_{l \in F} p_{ik}^{(t_1)} {}_{\tilde{F}} p_{kl}^{(t_2 - t_1)} \quad (i, j \in E), \tag{20}$$

$$= 0 \quad (i \in I,\ j \in \tilde{E}). \tag{21}$$

Corollary 4 . $p_i\{\tau_E(\omega) < \infty,\ x(\tau_E, \omega) = j,\ x(r, \omega) \in F,\ t_1 \leq r < \infty\}$

$$= \sum_{l \in \tilde{E}} \sum_{k \in F} \int_0^{t_1} {}_E p_{il}^{(t_1)} q_{lj} p_{jk}^{(t_1 - u)} U_{kF} du + \sum_{l \in \tilde{E}} \sum_{m \in \tilde{E} \cap F} \int_{t_1}^{\infty} {}_E p_{il}^{(t_1)} {}_{E \cup \tilde{F}} p_{lm}^{(u - t_1)} q_{mj} U_{jF}$$

$$du \quad (i \in \tilde{E},\ j \in E \cap F), \tag{22}$$

$$= \sum_{l \in \tilde{E}} \sum_{m \in F} \int_0^{t_1} {}_E p_{il}^{(t_1)} q_{lj} p_{jm}^{(t_1 - u)} U_{mF} du \quad (i \in \tilde{E},\ j \in E \cap \tilde{F}) \tag{23}$$

$$= \delta_{ij} \sum_{k \in F} p_{ik}^{(t_1)} U_{kF} \quad (i, j \in E) \tag{24}$$

$$= 0 \quad (i \in I,\ j \in \tilde{E}),$$

where $U_{kF} = \lim_{t \to \infty} \sum_{m \in F} {}_{\tilde{F}} p_{km}^{(t)}$.

Corollary 5 . For $0<t\leq t_1, p_i\{\tau_E(\omega)<t, x(\tau_E,\omega)=j, x(r,\omega)\in F, 0\leq r\leq t_1\}$

$$=\sum_{l\in\tilde{E}\cap F}\sum_{k\in F}\int_0^t {}_{\tilde{F}\cup E}p_{il}^{(u)}q_{lj}\,{}_{\tilde{F}}p_{ik}^{(t_1-u)}du \quad (i\in\tilde{E}\cap F, j\in E\cap F), \tag{27}$$

$$=\delta_{ij}\sum_{k\in F}p_{ik}^{(t_1)} \quad (i\in E\cap F, j\in E), \tag{28}$$

$$=0 \quad (i\in E, j\in\tilde{E}, \text{ or } i\in\tilde{E}\cap\tilde{F}, j\in I, \text{or } i\in\tilde{E}\cap F, j\in\tilde{E}\cup\tilde{F} \text{ or } i\in E\cap\tilde{F}, j\in E). \tag{29}$$

Proof: in the (11), we suppose that $t_1=0$ then we obtain (27). Thus corollary 5 is proved. ▌

Corollary 6. For $t_1<t$, $p_i\{\tau_E(\omega)<t, x(\tau_E,\omega)=j, x(r,\omega)\in F, 0\leq r\leq t_1\}$

$$=\sum_{l\in\tilde{E}\cap F}\sum_{k\in F}\int_0^{t_1} {}_{E\cup\tilde{F}}p_{il}^{(u)}q_{lj}\,p_{jk}^{(t_1-u)}du+\sum_{l\in\tilde{E}}\sum_{k\in\tilde{E}\cap F}\int_{t_1}^{t} {}_{E\cup\tilde{F}}p_{ik}^{(t_1)}\,{}_{E}p_{kl}^{(u-t_1)}q_{lj}du$$

$$(i\in\tilde{E}\cap F, j\in E\cap F), \tag{30}$$

$$=\sum_{l\in\tilde{E}}\sum_{k\in\tilde{E}\cap F}\int_{t_1}^{t} {}_{E\cup\tilde{F}}p_{ik}^{(t_1)}\,{}_{E}p_{kl}(u-t_1)q_{lj}du\,(i\in\tilde{E}\cap F, j\in E\cap\tilde{F}), \tag{31}$$

$$=\delta_{ij}\sum_{k\in F}p_{ik}^{(t_1)} \quad (i\in E\cap F, j\in E\cap\tilde{F} \text{ or } i\in E\cap F, j\in E\cap F), \tag{32}$$

$$=0 \quad (i\in\tilde{E}\cap F, j\in\tilde{E} \text{ or } i\in\tilde{E}\cap\tilde{F}, j\in I \text{ or } i\in E, j\in\tilde{E} \text{ or } i\in E\cap\tilde{F}, j\in E). \tag{33}$$

Proof: In (2), we suppose that $t_1=0$, then we obtain (30). Similary we may obtain (31) by (3), and corollary 6 is proved. ▌

Corollary 7. For $0<t\leq t_1$, $p_i\{\tau_E(\omega)<t, x(r,\omega)\in F, 0\leq r\leq t_1\}$

$$=\sum_{l\in\tilde{E}\cap F}\sum_{k\in F}\sum_{j\in E\cap F}\int_0^t {}_{E\cup\tilde{F}}p_{il}^{(u)}q_{lj}\,{}_{\tilde{F}}p_{jk}^{(t_1-u)}du \quad (i\in\tilde{E}\cap F), \tag{34}$$

$$=\sum_{k\in F}p_{ik}^{(t_1)} \quad (i\in E\cap F), \tag{35}$$

$$=0 \quad (i\in\tilde{F}). \tag{36}$$

Corollary 8 For $0<t_1<t$, $p_i\{\tau_E(\omega)<t, x(r,\omega)\in F, 0\leq r\leq t_1\}$

$$=\sum_{l\in\tilde{E}\cap F}\sum_{k\in F}\sum_{j\in E\cap F}\int_0^{t_1} {}_{E\cup\tilde{F}}p_{il}^{(u)}q_{lj}\,{}_{\tilde{F}}p_{jk}^{(t_1-u)}du$$

$$+\sum_{l\in\tilde{E}}\sum_{k\in\tilde{E}\cap F}\sum_{j\in E}\int_{t_1}^{t} {}_{E\cup\tilde{F}}p_{ik}^{(t_1)}\,{}_{E}p_{kl}^{(u-t_1)}q_{lj}du \quad (i\in\tilde{E}\cap F), \tag{37}$$

$$=\sum_{k\in F}p_{ik}^{(t_1)} \quad (i\in E\cap F), \tag{38}$$

$$=0 \quad (i\in\tilde{F}). \tag{39}$$

Corollary 9. $p_i\{\tau_E(\omega)<\infty,\ x(\tau_E,\omega)=j, x(r,\omega)\in F,\ 0\le r\le t_1\}$

$$=\sum_{l\in\tilde{E}}\sum_{k\in\tilde{E}\cap F}\int_{t_1}^{\infty}{}_{E\cup\tilde{F}}p_{ik}^{(t_1)}\,{}_Ep_{kl}^{(u-t_1)}q_{lj}\,du+\sum_{l\in E\cap\tilde{F}}\sum_{k\in\tilde{F}}\int_0^{t_1}{}_{E\cup\tilde{F}}p_{il}^{(u)}q_{lj}$$

$${}_{\tilde{F}}p_{jk}^{(t_1-u)}du\qquad(i\in\tilde{E}\cap F,\ j\in E\cap\tilde{F}).\tag{40}$$

$$=\sum_{l\in\tilde{E}}\sum_{k\in\tilde{E}\cap F}\int_{t_1}^{\infty}{}_{E\cup\tilde{F}}p_{ik}^{(u)}\,{}_Ep_{kl}^{(u-t_1)}q_{lj}du\quad(i\in\tilde{E}\cap F,\ j\in E\cap F),\tag{41}$$

$$=\delta_{ij}\sum_{k\in F}p_{ik}^{(t_1)}\qquad(i\in E\cap F,\ j\in E),\tag{42}$$

$$=0\quad(i\in E,\ j\in\tilde{E}\text{ or }i\in\tilde{E}\cap\tilde{F},\ j\in I\text{ or }i\in\tilde{E}\cap F,\ j\in\tilde{E}\text{ or }i\in E\cap\tilde{F},\ j\in E).\tag{43}$$

Corollary 10. $p_i\{\tau_E(\omega)<\infty,\ x(\tau_E,\omega)=j, x(r,\omega)\in F,\ 0\le r<\infty\}$

$$=\sum_{l\in\tilde{E}\cap F}\int_0^{\infty}{}_{E\cup\tilde{F}}p_{il}(u)q_{lj}U_{jF}\,du\qquad(i\in\tilde{E}\cap F,\ j\in E\cap F),\tag{44}$$

$$=\delta_{ij}U_{iF}\qquad(i\in E\cap F,\ j\in E),\tag{45}$$

$$=0\quad(i\in\tilde{E}\cap F,\ j\in\tilde{E}\cup\tilde{F}\text{ or }i\in E,\ j\in\tilde{E}\text{ or }i\in\tilde{E}\cap\tilde{F},\ j\in I,\text{or }i\in E\cap\tilde{F},\ j\in E).\tag{46}$$

Proof: We suppose that $t_1=0$ in (22), then we obtain (44), and corollary 10 is proved. ▌

Corollary 11. For $t_2<t$, $p_i\{\tau_E(\omega)<t,\ x(r,\omega)\ F,\ t_1\le r\le t_2\}$

$$=\sum_{l\in\tilde{E}}\sum_{m\in F}\sum_{k\in F}\sum_{j\in E}\int_0^{t_1}{}_Ep_{il}^{(u)}q_{lj}p_{jm}^{(t_1-u)}\,{}_{\tilde{F}}p_{mk}^{(t_2-t_1)}du$$

$$+\sum_{l\in\tilde{E}}\sum_{m\in\tilde{E}\cap F}\sum_{k\in F}\sum_{j\in E\cap F}\int_{t_1}^{t_2}{}_Ep_{il}^{(t_1)}\,{}_{E\cup\tilde{F}}p_{lm}^{(u-t_1)}q_{mj}\,{}_{\tilde{F}}p_{jk}^{(t_2-u)}du$$

$$+\sum_{l\in\tilde{E}}\sum_{m\in\tilde{E}\cap F}\sum_{k\in\tilde{E}}\sum_{j\in E}\int_{t_2}^{t}{}_Ep_{il}^{(t_1)}\,{}_{E\cup\tilde{F}}p_{lm}^{(t_2-t_1)}\,{}_Ep_{mk}^{(u+t_1-t_2)}q_{kj}du\quad(i\in\tilde{E})\tag{47}$$

$$=\sum_{k\in F}\sum_{l\in F}p_{ik}^{(t_1)}\,{}_{\tilde{F}}p_{kl}^{(t_2-t_1)}\qquad(i\in E).\tag{48}$$

Corollary 12. For $t_1<t\le t_2$, $p_i\{\tau_E(\omega)<t, x(r,\omega)\in F,\ t_1\le r\le t_2\}$

$$=\sum_{l\in\tilde{E}}\sum_{m\in F}\sum_{k\in F}\sum_{j\in E}\int_0^{t_1}{}_Ep_{il}^{(u)}q_{lj}p_{jm}^{(t_1-u)}\,{}_{\tilde{F}}p_{mk}^{(t_2-t_1)}du$$

$$+\sum_{l\in\tilde{E}}\sum_{m\in\tilde{E}\cap F}\sum_{k\in F}\sum_{j\in E\cap F}\int_{t_1}^{t}{}_Ep_{il}^{(t_1)}\,{}_{E\cup\tilde{F}}p_{lm}^{(u-t_1)}q_{mj}\,{}_{\tilde{F}}p_{jk}^{(t_2-u)}du\quad(i\in\tilde{E})\tag{49}$$

$$= \sum_{k\in F} \sum_{l\in F} p_{ik}^{(t_1)}\ {}_{\tilde{F}}p_{kl}^{(t_2-t_1)} \quad (i\in E). \tag{50}$$

Corollary 13. For $0<t\leq t_1$, $p_i\{\tau_E(\omega)<t,\ x(r,\omega)\in F,\ t_1\leq r\leq t_2\}$

$$= \sum_{l\in\tilde{E}} \sum_{m\in F} \sum_{k\in F} \sum_{j\in E} \int_0^t {}_E p_{il}^{(u)} q_{lj}\, p_{jm}^{(t_1-u)}\ {}_{\tilde{F}}p_{mk}^{(t_2-t_1)} du \quad (t\in\tilde{E}), \tag{51}$$

$$= \sum_{l\in F} \sum_{k\in F} p_{ik}^{(t_1)}\ {}_{\tilde{F}}p_{kl}^{(t_2-t_1)} \quad (i\in E). \tag{52}$$

Corollary 14. $p_i\{\tau_E(\omega)<\infty,\ x(r,\omega)\in F,\ t_1\leq r<\infty\}$

$$= \sum_{l\in\tilde{E}} \sum_{m\in F} \sum_{j\in E} \int_0^{t_1} {}_E p_{il}^{(t_1)} q_{lj}\, p_{jm}^{(t_1-u)} U_{mF} du + \sum_{l\in\tilde{E}} \sum_{m\in\tilde{E}\cap F} \sum_{j\in E\cap F} \int_{t_1}^{\infty} {}_E p_{il}^{(t_1)}\ {}_{E\cup\tilde{F}}p_{lm}^{(u-t_1)}.$$

$$\cdot q_{mj} U_{jF}\, du \quad (i\in\tilde{E}), \tag{53}$$

$$= \sum_{k\in F} p_{ik}^{(t_1)} U_{kF} \quad (i\in E). \tag{54}$$

Let $M(t_0,\omega)=\sup\{x(r,\omega): 0\leq r\leq t_0\}$ (55)

$$\tau_{M(t_0,\omega)} = \begin{cases} \inf\{t: 0\leq t\leq t_0,\ x(t,\omega)=M(t_0,\omega)\}, & \text{if it is nonempty}, \\ =\infty, & \text{otherwise}. \end{cases} \tag{56}$$

$M(t_0,\omega)$ is called maximum play distance of $X(\omega)$ before time t_0. $\tau_{M(t_0,\omega)}$ is called the first hitting time of $M(t_0,\omega)$.

Theorem 2. $p_i\{M(t_0,\omega)=j\}$

$$= \sum_{l\in\{0-j-1\}} \sum_{k\in\{0-j\}} \int_0^{t_0} {}_{\{j-\}}p_{il}^{(u)}\, q_{lj\{j+1-\}}\, p_{jk}^{(t_0-u)} du \quad (i\in\{0-j-1\}) \tag{57}$$

$$= \sum_{k\in\{0-j\}} {}_{\{j+1-\}}p_{ik}^{(t_0)} \quad (i=j) \tag{58}$$

$$=0 \quad i\in\{j+1-\}, \tag{59}$$

where $\{i-j\}=\{i,i+1,i+2,\cdots j\}$, $\{j-\}=\{j,j+1,j+2,\cdots\}$.

Proof: We prove that

$$\{\omega: M(t_0,\omega)=j\} \doteq \{\omega: \tau_{\{j\}}(\omega)\leq t_0,\ x(t,\omega)\in\{0-j\},\ 0\leq t\leq t_0\}, \tag{60}$$

If $\omega\in\{\omega: M(t_0,\omega)=j\}$, then for arbitrary $0<\varepsilon<1$, there is a $t\in[0,t_0]$ such that $|x(t,\omega)-j|<\varepsilon$. Hence $x(t,\omega)=j$, and

$\omega\in\{\omega: \tau_{\{j\}}(\omega)\leq t_0,\ x(t,\omega)\in\{0-j\}\ 0\leq t\leq t_0\}$,

$$\{\omega: M(t_0,\omega)=j\}\subset\{\omega: \tau_{\{j\}}(\omega)\leq t_0,\ x(t,\omega)\in\{0-j\},\ 0\leq t\leq t_0\}. \tag{61}$$

Conversely, if $\omega\in\{\omega: \tau_{\{j\}}(\omega)<t_0,\ x(t,\omega)\in\{0-j\},\ 0\leq t\leq t_0\}$, by right lower

semicontinuity, it is clear that $\omega \in \{\omega: M(t_0, \omega) = j\}$, and

$$\{\omega: \tau_{\{j\}}(\omega) < t_0,\ x(t,\omega) \in \{0-j\}, 0 \le t \le t_0\} \subset \{\omega: M(t_0,\omega) = j\}. \quad (62)$$

Furthermore, $\{\omega: \tau_{\{i\}}(\omega) < t_0,\ x(t,\omega) \in \{0-j\}, 0 \le t \le t_0\} \doteq \{\omega: \tau_{\{j\}}(\omega) \le t_0,\ x(t,\omega) \in \{0-j\}, 0 \le t < t_0\}$. (63)

Hence (60) is proved by (61)-(63).

Next, using corollary 5 of theorem 1, we have

$$p_i\{M(t_0, \omega) = j\} = p_i\{\tau_{\{j\}}(\omega) \le t_0,\ x(t,\omega) \in \{0-j\},\ 0 \le t \le t_0\}$$

$$= \sum_{l \in \{\tilde{j}\} \cap \{0-j\}} \sum_{k \in \{0-j\}} \int_0^{t_0} {}_{\widetilde{\{0-j\}} \cup \{j\}} p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t_0-u)} du$$

$$= \sum_{l \in \{0-j-1\}} \sum_{k \in \{0-j\}} \int_0^{t_0} {}_{\{j-\}} p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t_0-u)} du \quad (i \in \{0-j-1\}).$$

Then theorem2 is proved. ▮

Corollary 1. Let $E = \{j_1, j_2, j_3, \cdots\}$ where $j_1 < j_2 < j_3 < \cdots$, $E(j_n) = \{j_n, j_{n+1}, j_{n+2}, \cdots\}$, then $p_i\{M(t_0,\omega) \in E\}$

$$= \sum_{j \in E(j_n)} \sum_{l \in \{0-j-1\}} \sum_{k \in \{0-j\}} \int_0^{t_0} {}_{\{j-\}} p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t_0-u)} du + \sum_{k \in \{0-j_{n-1}\}} {}_{\{j_{n-1}+1-\}} p_{ik}^{(t_0)} \cdot \delta_{ij_{n-1}} \quad (i \in \{j_{n+1} - j_n^{-1}\}, \text{ for } n = 1,2,3,\cdots,\ j_0 = 0)$$ (64)

$$= 0 \text{ for other } i. \quad (65)$$

Theorem 3. For $t \le t_0$,

$$p_i\{\tau_{M(t_0,\omega)} < t, x(\tau_{M(t_0,\omega)}, \omega) = j\}$$

$$= \sum_{l \in \{0-j-1\}} \sum_{k \in \{0-j\}} \int_0^{t} {}_{\{j-\}} p_{il}^{(u)} q_{lj\{j+1-\}} p_{ik}^{(t_0-u)} du \quad (i \in \{0-j-1\} \quad (66)$$

$$= \sum_{k \in \{0-j\}} {}_{\{j+1-\}} p_{ik}^{(t_0)} \quad (i = j) \quad (67)$$

$$= 0 \quad (i \in \{j+1-\}). \quad (68)$$

Proof: Using right lower semicontinuity and corollary 5 of theorem 1, we have $p_i\{\tau_{M(t_0,\omega)} < t, x(\tau_{M(t_0,\omega)}, \omega) = j\}$

$$= \sum_{k \in \{0-\}} p_i\{\tau_{M(t_0,\omega)} < t, x(\tau_{M(t_0,\omega)}, \omega) = j, M(t_0,\omega) = k\}$$

$$= p_i\{\tau_{\{j\}} < t, x(\tau_{\{j\}}, \omega) = j,\ M(t_0, \omega) = j\}$$

$$= p_i\{\tau_{\{j\}}(\omega) < t, x(r,\omega) \in \{0-j\},\ 0 \le r \le t_0\}.$$

$$= \sum_{l\in\{0-j-1\}} \sum_{k\in\{0-j\}} \int_0^t {}_{\{j-\}}p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t_0-u)} du \quad (i\in\{0-j-1\}). \tag{69}$$

Herce (66) is proved by (69). ▌

Corollary 1. For $0<t\le t_0$, $p_i\{\tau_{M(t_0,\omega)}<t, M(t_0,\omega)\in\{0-\}\}$

$$= \sum_{j\in E(n+1)} \sum_{l\in\{0-j-1\}} \sum_{k\in\{0-j\}} \int_0^t {}_{\{j-\}}p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t_0-u)} du$$

$$+ \sum_{k\in\{0-n\}} {}_{\{n+1-\}}p_{ik}^{(t_0)} \quad (i=n,\ n=0,1,2,\cdots). \tag{70}$$

Theorem 4. If $F(t)=\{\omega: \lim_{s\uparrow u} x(s,\omega)=\infty$, for some one u, $\lim_{s\uparrow v} x(s,\omega)$

$<\infty$ for every v, $0\le v<u\le t\}$, (71)

then $F(t)=\{\omega: M(t,\omega)=\infty\}$, (72)

$$p_i\{F(t)\} = 1 - \sum_{i\in E(n)} \sum_{l\in\{0-j-1\}} \sum_{k\in\{0-j\}} \cdot \int_0^t {}_{\{j-\}}p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t-u)} du$$

$$- \sum_{k\in\{0-n-1\}} {}_{\{n-\}}p_{ik}^{(t)} \quad (i=n-1, n=1,2,\cdots). \tag{73}$$

Proof: it is clear that $F(t)\subset\{\omega: M(t,\omega)=\infty\}$. (74)

Conversly, if $\omega\in\{\omega: M(t,\omega)=\infty\}\cap\{\omega: \lim_{s\uparrow u} x(s,\omega)=\infty=x(u,\omega)$ for some one $0\le u\le t\}$, then there is a sequence $\{t_n: n=1,2,3,\cdots\}\subset[0,t]$ such that

$$\lim_{n\to\infty} x(t_n,\omega)=\infty.$$

If $\{t_n: n=1,2,3,\cdots\}$ is a finite set then there is a u $\in\{t_n: n=1,2,3,\cdots\}$ such that $t_{n_l}=u$ for $l=1,2,3,\cdots$ Hence $x(u,\omega)=\infty$ and $\{s: x(s,\omega)=\infty,\ 0\le s\le t\}$ is nonempty. Further there is $u_0=\inf\{s: x(s,\omega)=\infty,\ 0\le s\le t\}$. If u_0 is a limit point, then by right lower semicontinuity we know that $x(u_0,\omega)=\infty$. If $\lim_{s\uparrow u_0} x(s,\omega)\neq\infty$, then $\lim_{s\uparrow u_0} x(s,\omega)=j$, where the ω belonges to a null set. Since $\omega\in\Omega_0\cap\{\omega: M(t,\omega)=\infty\}\cap\{\omega: \lim_{s\uparrow u} x(s,\omega)=\infty=x(u,\omega)$ for some one $0\le u\le t\}$, we have $\lim_{s\uparrow u_0} x(s,\omega)=\infty$ $\lim_{s\uparrow v} x(s,\omega)<\infty$, $0\le v<u_0\le t$ for some one u_0 ard every $0\le v<u_0$. Further more $\omega\in F(t)$. If u_0 is not a limit point, we have $\omega\in F(t)$ too. Namely, $\{\omega: M(t,\omega)=\infty\}\subset F(t)$. (75)

If $\{t_n: n=1,2,3,\cdots\}$ is an infinite set, then there is a subsequence $\{t_{n_l}: l=1,2,3,\cdots\}$ such that $\lim_{l\to\infty} t_{n_l}=u\in[0,t]$. If $\lim_{l\to\infty} x(t_{n_l},\omega)=\infty$ and $x(u,\omega)=j$, then the ω belonges to a null set. thus (75) is true, where Ω_0 is a sure event. Hence (12) is hold.

Next, $\{\omega: M(t,\omega)=\infty\} = \Omega \setminus (\bigcup_{j\in I} \{\omega: M(t,\omega)=j\})$, $p_i\{F(t)\}=1-\sum_{j\in I}$ $p_i\{M(t,\omega)=j\}$, and (73) is proved. ▌

Suppose that $B(t)=\{\omega$: Markov chains is not reaching ∞ in $[0,t]\}$.

$B(\infty)=\{\omega$: Markov chains is not reaching ∞ in $[0,\infty)\}$,

$F(\infty)=\{\omega$: Markov chains is first reaching ∞ in time u, $0\le u<\infty\}$,

then $B(\infty)=\bigcap_{t=1}^{\infty} B(t)$, $F(\infty)=\bigcup_{t=1}^{\infty} F(t)$, $B(t)\cup F(t)=B(\infty)\cup F(\infty)=\Omega$.

Corollary 1. $p_i\{B(t)\}=\sum_{j\in E(n)}\sum_{l\in\{0-j-1\}}\sum_{k\in\{0-j\}}\int_0^t {}_{\{j-\}}p_{il}^{(u)} q_{lj\{j+1-\}}$ $p_{jk}^{(t-u)}du + \sum_{k\in\{0-n-1\}} {}_{\{n-\}}p_{ik}^{(t)}$ $(i=n-1, n=1,2,3,\cdots)$, (76)

$$p_i\{B(\infty)\}=\lim_{t\to\infty}\sum_{j\in E(n)}\sum_{l\in\{0-j-1\}}\sum_{k\in\{0-j\}}\int_0^t {}_{\{j-\}}p_{ij}^{(u)} q_{lj\{j+1-\}} p_{ik}^{(t-u)}du + U_{i\{0-n-1\}} \quad (i=n-1, n=1,2,3,\cdots), \tag{77}$$

$$p_i\{F(\infty)\}=1-\lim_{t\to\infty}\sum_{j\in E(n)}\sum_{l\in\{0-j-1\}}\sum_{k\in\{0-j\}}\int_0^t {}_{\{j-\}}p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t-u)}du - U_{i\{0-n-1\}} \quad (i=n-1, n=1,2,3,\cdots). \tag{78}$$

Corollary 2. $\{p_{ij}(t): t\ge 0, i, j\in I\}$ is the minimal processes if and only if for $i=n-1, n=1,2,3,\cdots$

$$\lim_{t\to\infty}\sum_{j\in E(n)}\sum_{l\in\{0-j-1\}}\sum_{k\in\{0-j\}}\int_0^t {}_{\{j-\}}p_{il}^{(u)} q_{lj\{j+1-\}} p_{jk}^{(t-u)}du = 1-U_{i\{0-n-1\}} \tag{79}$$

Corollary 3. If for some $i=n-1$

$$\lim_{t\to\infty}\sum_{j\in E(n)}\sum_{l\in\{0-j-1\}}\sum_{k\in\{0-j\}}\int_0^t {}_{\{j-\}}p_{il}^{(u)} q_{lj\{j+1-\}} p_{ik}^{(t-u)}du < 1-U_{i\{0-n-1\}}, \tag{80}$$

then $\{p_{ij}(t): t\ge 0\ i, j\in I\}$ is not a minimal processes.

Let $M(\omega)=\sup\{x(r,\omega): 0\le r<\infty\}$, (81)

$$\tau_{M(\omega)} = \begin{cases} \inf\{t: t\ge 0, x(t,\omega)=M(\omega)\}, & \text{if it is nonempty,} \\ \infty, & \text{otherwise.} \end{cases} \tag{82}$$

$M(\omega)$ is called maximum play distance of $X(\omega)$ in the finite time. $\tau_{M(\omega)}$ is called the first hitting time of $M(\omega)$.

Theorem 5. $p_i\{M(\omega)=j\}$

$$=\sum_{l\in\{0-j-1\}}\int_0^\infty {}_{\{j-\}}p_{il}^{(u)}q_{lj}U_{j\{0-j\}}\,du \quad (i\in\{0-j-1\}) \tag{83}$$

$$=U_{i\{0-j\}} \quad (i=j) \tag{84}$$

$$=0 \quad (i\in\{j+1-\}). \tag{85}$$

Proof: Using corollary 10 of theorem 1,we may prove theorem 5. ▮

Corollary 1. Let $E=\{j_1,j_2,j_3\cdots\}$, where $j_1<j_2<j_3<\cdots$, then $p_i\{M(\omega)\in E\}$

$$=\sum_{j\in E(j_n)}\sum_{l\in\{0-j-1\}}\int_0^\infty {}_{\{j+\}}p_{il}^{(u)}q_{lj}U_{j\{0-j\}}du+U_{j_{n-1}\{0-j_{n-1}\}}\cdot\delta_{ij_{n-1}} \tag{86}$$

$(i\in\{j_{n-1}-j_n-1\},\ n=1,2,3,\cdots,\ j_0=0)$,

$$=0 \quad \text{for other } i. \tag{87}$$

Theorem 6. For $0<t\le\infty$, $p_i\{\tau_{M(\omega)}<t,\ x(\tau_{M(\omega)},\omega)=j\}$

$$=\sum_{l\in\{0-j-1\}}\int_0^t {}_{\{j-\}}p_{il}^{(u)}q_{lj}U_{j\{0-j\}}du \quad (i\in\{0-j-1\}), \tag{88}$$

$$=U_{i\{0-j\}} \quad (i=j), \tag{89}$$

$$=0 \quad (i\in\{j+1-\}). \tag{90}$$

Proof: Using corollary 7 of theorem 1 and dominant convergence theorem, we may prove theorem 6. ▮

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