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A New Approach to The Upwind Finite Elements*

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We want to describe a new way for deriving the upwind finite elements by a simple example. Consider the boundary value problem,

$$\begin{cases} -\varepsilon u'' + u' = 1, & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

with ε a small parameter. Let $h = 1/N$ and $x_i = ih$, $0 \le i \le N$. Denote

$$v_i(x) = \begin{cases} (x - x_{i-1})/h, & x_{i-1} < x \le x_i \\ (x_{i+1} - x)/h, & x_i < x \le x_{i+1}, \quad 1 \le i \le N-1. \\ 0 & \text{otherwise} \end{cases}$$

And set V_h the space spanned by $v_1, \cdots, v_{N-1}\}$. The normal finite element method for problem (1) is to find $u_h \in V_h$, such that,

$$\varepsilon \int_0^1 u_h' v_h' \mathrm{d}x - \int_0^1 u_h v_h' \mathrm{d}x = \int_0^1 v_h \mathrm{d}x, \quad \forall v_h \in V_h. \tag{2}$$

Let $u_h = \sum_{i=1}^{N-1} u_i v_i$, $u_0 = u_N = 0$, and $\tilde{u}_h(x) = ((1-a)u_j + (1+a)u_{j-1})/2$ for $x \in [x_{j-1}, x_j)$ with a a constant to be determined. Now we replace u_h in the second term of the left side of equation (2) by $\tilde{u}_h$, and get

$$\begin{cases} \left(-\frac{\varepsilon}{h^2} + \frac{(1-a)}{2h}\right) u_{i+1} + \left(\frac{2\varepsilon}{h^2} + \frac{a}{h}\right) u_i + \left(-\frac{\varepsilon}{h^2} - \frac{(1+a)}{2h}\right) u_{i-1} = 1, \\ \qquad\qquad 1 \le i \le N-1 \\ u_0 = u_N = 0. \end{cases} \tag{3}$$

Equations (3) is the normal finite element with $a = 0$, and the upwind finite element with $a = 1$. The above method need not introduce a special test functions as in [1]. The upwind meaning can be seen from $\tilde{u}_h$ directly.

This method can be generalized to other cases. For example, we can get the upwind defference scheme of five points in the case of two dimension.

Reference

[1] Zienkiewicz. O.C.. Heinrich, J.C.. Finite Elements in Fluids, Vol. 3, Wiley, London, 1978.

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