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Fixed Point Theorems for Mappings in Compact Menger Space*

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Suppose that $(X, \mathscr{F}, \Delta)$ is a Menger space, t-norm Δ satisfies $\sup_{x<1} \Delta(x, a) = a$ for all $a \in [0, 1]$. From the remark in [1, p.330] we know that $(X, \mathscr{F}, \Delta)$ is a Hausdorff space in the topology $\mathscr{T}$ induced by the family of neighborhoods

$$\{\{x \in X; F_{xp}(\varepsilon) > 1-\lambda\},\ p \in X,\ \varepsilon, \lambda > 0\}$$

according to general method in topology, we can define the notion, such as $\mathscr{T}$-convergence, $\mathscr{T}$-Cauchy sequence, $\mathscr{T}$-continuous self-mappings of X and $(X, \mathscr{F}, \Delta)$ is $\mathscr{T}$-complete, $\mathscr{T}$-self-seqential compact, $\mathscr{T}$-compact etc [1-3].

Lemma 1 [2] $A \subset X$ being a $\mathscr{T}$-self-sequential compact subset is equivalent to $A \subset X$ being a $\mathscr{T}$-compact subset

Lemma 2 Let $A, B \subset X$ be $\mathscr{T}$-compact subsets, then for all $t \geq 0$ there exists $p_0(t) \in A$ and $q_0(t) \in B$ such that

$$\inf_{p \in A, q \in B} F_{pq}(t) = F_{p_0(t)q_0(t)}(t)$$

Throughout this paper we always assume that $(X, \ \ , \Delta)$ is a $\mathscr{T}$-compact Menger space, t-norm Δ satisfies $\sup_{x<1} \Delta(x, a) = a$ for all $a \in [0, 1]$. and T, S are $\mathscr{T}$-continuous self-mappings of X.

The main results of this paper is as follows:

Theorem 1 Let m and n be two nonnegative integral numbers.

$$F_{T^m x S^n y}(t) > \inf_{p \in \{T^k x\}_{k=0}^{\infty}, q \in \{S^k y\}_{k=0}^{\infty}} F_{pq}(t)$$

for all $x, y \in X$ and $t > 0$ which makes on the right-hand less than 1 holds. Then

(i) T and S have an unique common fixed point x_*, and x_* is an unique fixed point of T and S.

(ii) for each $x \in X$ the sequences of iteration $T^n x \xrightarrow{\mathscr{T}} x_*$ and $S^n x \xrightarrow{\mathscr{T}} x_*$.

Theorem 2 Suppose that t-norm Δ satisfies $\Delta(a, b) \geq \max\{a+b-1, 0\}$ for all $a, b \in [0, 1]$, and there exist $n, m: X \to Z^+$ (the set of all positive integers) such that

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$$F_{T^{n(x)}x\,S^{m(y)}y}(t) > \inf\{F_{xy}(t),\ F_{xT^{n(x)}x}(t),\ F_{yS^{m(y)}y}(t)\}$$

for all $x, y \in X$ and $t > 0$ which makes on the right-hand less than 1 holds. Suppose further that

1° $n(x)|n(Tx)$ and $m(y)|m(Sy)$ for all $x, y \in X$;

2° $T^{n(x_n)}x_n \xrightarrow{\mathscr{T}} T^{n(x)}x$ and $S^{m(x_n)}x_n \xrightarrow{\mathscr{T}} S^{m(x)}x$ for any $x_n \xrightarrow{\mathscr{T}} x$.

Then the conclusion (i) of Theorem 1 still holds.

From Theorem 2, we can easily deduce the following corollary.

Corollary Let t-norm Δ be the same as Theorem 2. Suppose that there exist $n, m \in Z^{+}$ such that

$$F_{T^n x S^m y}(t) > \inf\{F_{xy}(t),\ F_{xT^n x}(t),\ F_{yS^m y}(t)\}$$

for all $x, y \in X$ and $t > 0$ which makes on the-right-hand less than 1 holds. Then the conclusion of Theorem 2 still holds.

Remark Above results are easily extended to a family of mappings. Using the same argument as Theorem 2 and 3, we can change many fixed point theorems in compact metric space (e.g. see [3]) into the fixed point theorems in $\mathscr{T}$-compact Menger space.

References

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