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A Note on Young's Integral Inequality*

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The well-known inequality of W.H.Young may be written as

$$ab \le \int_0^a \phi(x)\,dx + \int_0^b \psi(x)\,dx,$$

where $a>0$, $b>0$, and $\phi(x) \in C(0,\infty)$ increases strictly with x and $\phi(0)=0$, and $\psi(x)$ is the inverse function so that $\psi(\phi(x)) = \phi(\psi(x)) = x$. An investigation into the graphs of the functions $y=\phi(x)$ and $x=\psi(y)$ reveals that

$$\int_0^a \phi(x)\,dx + \int_0^b \psi(x)\,dx = b\psi(b) - \int_0^{\psi(b)} \phi(x)\,dx. \qquad (*)$$

This holds generally when $a<\psi(b)$ or $a \ge \psi(b)$. Since $[\psi(b)-a][b-\phi(a)] \ge 0$ it is easy to deduce from $(*)$ the inequalities

$$ab \le \int_0^a \phi\,dx + \int_0^b \psi\,dx \le a\phi(a) + b\psi(b) - \phi(a)\psi(b). \qquad (1)$$

Assuming the convexity of $\phi(x)$, one may get a refinement of (1), viz.

$$ab + \frac{1}{2}[b-\phi(a)][\psi(b)-a] \le \int_0^a \phi\,dx + \int_0^b \psi\,dx, \qquad (2)$$

whenever $\phi''(x)[b-\phi(a)] \ge 0$. The inequality will be reversed if $\phi''(x)[b-\phi(a)] \le 0$.

A further refinement of (2) can be obtained via $(*)$ and by making use of a known proposition in Polya-Szegö's "Aufgaben und Lehrsatze aus der Analysis". **Theorem**: Let $\phi'(x)$ be monotone and denote

$$S_n = S_n(a,b) = b\psi(b) - h\left[\frac{\phi(a)+b}{2} + \sum_{j=1}^{n-1}\phi(a+jh)\right], \qquad (3)$$

where $h=(\psi(b)-a)/n$, $(n=2,3,\cdots)$. Then we have

$$\left| \int_0^a \phi\,dx + \int_0^b \psi\,dx - S_n \right| \le \frac{1}{8}h^2 \left|\phi'(a) - \phi'(\psi(b))\right|.$$

In particular, for n large $S_n(0,b)$ provides an approximation to $\int \psi\,dx$ with error estimate $O(n^{-2})$.

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