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On Development of a Ky Fan Inequality of the Complementary A-G Type*

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Abstract Two new proofs of a discrete Ky Fan inequality of the complementary A-G type are given. Its continuous version and determinantal analogue on a set of pairwise commutative positive definite matrices are established. A further extension concerning general positive definite matrices of the inequality is also suggested.

I. Introduction

In the early sixties, Ky Fan established a remarkable and interesting inequality relating the A-G (abbreviation of arithmetic and geometric) means in a complementary manner, now known as a Ky Fan inequality of the complementary A-G type (cf. [22,], [23]), which was first recorded in Beckenbach and Bellman [3, p.5] (see also [13, p.363]):

For $x_j \in (0, \frac{1}{2}]$, $j=1,\cdots,n$, we have

$$(\Pi x_j)/(\Sigma x_j)^n \leqslant (\Pi(1-x_j))/(\Sigma(1-x_j))^n \qquad (1)$$

with equality iff all x_j are equal.

Here and in what follows Σ and Π are used to designate $\Sigma_{j=1}^n$ and $\Pi_{j=1}^n$, respectively, whenever confusion is unlikely to occur.

In 1964, Levinson [9] generalized inequality (1) to an inequality involving a nonnegative thrice differentiable function. Along the same line of Levinson, further generalizations and variants can be found in Bullen [4], Chan, Goldberg and Gonek [5], Lawrence and Segalman [8], Popoviciu [15], and, Vasić and Janić [18].

Recently, Wang ([22], [23]) extended (1) to the case of unequal weights as follows:

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$$\Pi(x_j/(1-x_j))^{a_j} \leqslant A/(1-A), \qquad (2)$$

where $x_j \in [0, \frac{1}{2}]$, $a_j > 0, \Sigma a_j = 1$, $A = \Sigma a_j x_j$, $j = 1, \cdots, n$, with equality iff all x_j are equal. Then, he established (2) respectively in [22] by introducing an inequality of Rado type associated with (2) and in [23] through the functional equation approach of dynamic programming (e.g. see [3, p. 6]).

It should also be noted that the original unpublished forward and backward induction proof (cf [3]) and a usual induction proof of (1) are given in Ozeki [14, p. 128, p. 60], while a proof of (1) by a Schur-convex function can be found in Marshall and Olkin [11, p. 98].

In order to unify our subsequent arguments, we shall use an equivalent form of (2) as follows:

$$\Sigma a_j[\ln x_j - \ln(1-x_j)] \leqslant \ln A - \ln(1-A) \qquad (3)$$

where $x_j \in (0, \frac{1}{2}]$, etc. as above.

In this paper, we shall not only add two new proofs of (3) (and a fortiori (2) and (1)) to the collection displayed above, but also establish its continuous version and determinantal anologue on a set of pairwise commutative positive definite matrices.

To this end, in section 2, we shall summarize notations and definitions that will be used. In section 3, we shall present theorems concerning concavity of functions as well as nonlinear functionals. In the following sections, we shall successively establish the aforementioned results by means of the theorems provided in section 2, and conclude with some remarks and a conjecture concerning the Ky Fan inequality of the complementary **A-G** type on a general set of positive definite matrices.

2. Notations and definitions

Let us begin by displaying some notations and symbols that we shall need.

$\mathbf{R}$ = the field of real numbers.

$\mathbf{R}^+ = \{x \in \mathbf{R} \mid x > 0\}$, $\mathbf{I} = (0, \frac{1}{2}]$.

$\mathbf{R}^n = \{(x_1, \cdots, x_n) \mid x_j \in \mathbf{R}, \ j = 1, \cdots, n\}$.

Ω = an open convex subset of $\mathbf{R}^n$ (or $\mathbf{I}^n$).

$\mathbf{X} = \{f \mid f: \Omega \to \mathbf{I}\}$ = the set of integrable function on Ω.

U = an open convex subset of X.

F = $\{F \mid F: \mathrm{U} \to \mathbf{R}^+\}$ = the set of functionals on U.

$\langle x, y\rangle$ = the inner product of x and y, $x, y \in \mathbf{R}^n$.

$f_j = \partial t/\partial x_j$, $\nabla f(x) = (f_1(x), \cdots, f_n(x))$, $x \in \mathbf{R}^n$.

$$f_{jk}=\partial^2 f/\partial x_k \partial x_j,\quad x\in \mathbf{R}^n.$$
$$Q=\langle f_{jk}(x)u,u\rangle=\sum_{j,k} f_{jk}u_j u_k.$$
$$\langle f,g\rangle=\int fg\,d\mu,\quad f,g\in X.$$

Here and in what follows $\int$ and $d\mu$ are used to designate $\int_\Omega$ and $d\mu(x)=dx_1\cdots dx_n$ respectively, whenever confusion is unlikely to occur.

$$\phi(\lambda)=F(f+\lambda h),\quad F\in \mathrm{F},\ f,h\in \mathrm{X},\ \lambda\in \mathbf{R}.$$

$$\phi'(0)=\frac{d}{d\lambda}F(f+\lambda h)\Big|_{\lambda=0}=\langle \partial_f F,h\rangle=\langle \partial F,h\rangle,$$

$$\phi''(0)=\frac{d^2}{d\lambda^2}F(f+\lambda h)\Big|_{\lambda=0}=\langle \partial_f^2 F,h^2\rangle=\langle \partial^2 F,h^2\rangle=Q,$$

where $\phi'(0)$ and $\phi''(0)$ are said to be the first and second Gateaux differentials (e.g.sec [17; p.35]) respectively.

Definition 1 The functional A: F→R is affine if for every $f\in$ F, $A(f)=L(f)+b$ where L: F→R is linear and b is a constant in R.

Definition 2 A concave functional F: U→R has support at $g\in$ U if there is an affine functional A: F→R such that $A(g)=F(g)$ and $A(f)\geqslant F(f)$ for every $f\in$ U.

Similat definitions concerning the support of a concave function on $\mathbf{R}^n$ are typical. (cf. Roberts and Varberg [14, p.108] for details).

3. Theorems concerning concavity

In this section, we state a simple concavity theorem as Theorem 1, and cite Theorem F on page 103 and Theorem B on page 108 of [12] with some modifications as Theorems 2 and 3 respectively. Then, corresponding to Theorems 2 and 3, we adopt the idea of Vainberg [17] and present Theorems 4 and 5 concerning the concavity of nonlinear positive functionals for our purpose (see also [21],[24]).

Theorem 1 Let $\eta(x)=\ln x-\ln(1-x)$ be defined on I. Then η is concave on I.

Theorem 2 Let f hava continuous second partial derivatives f_{jk} throughout an open convex set Ω of $\mathbf{R}^n$. Then f is concave iff the quadratic form Q is nonpositive of all x, $u\in\Omega$. Moreover, if Q is negative on Ω, then f is strictly concave.

Theorem 3 The function f is concave on Ω iff f has support $A(x)$ at each point $x_0\in\Omega$, $A(x_0)=f(x_0)$ and $A(x)\geqslant f(x)$ for every $x\in\Omega$.

Theorem 4 Let F have continuous second Gateaux differential on U. Then F is concave iff the quadratic form Q is nonpositive for all $f\in$ U, $h\in$ F. Moreover, if Q is negative on U, then F is strictly concave.

Theorem 5 The nonlinear positive functional F is concave on U iff F has support $A(f)$ at each point $\xi\in U$, $A(\xi)=F(\xi)$ and $A(f)\geq F(f)$ for every $f\in U$.

4. Two new proofs

In order to give two new proofs of inequality (1) (or (2)), we establish instead its equivalent inequality (3) as follows. First, since $\eta''(x)=-(1-2x)/x^2(1-x)^2<0$ on I, the concavity of Theorem 1 is established, which in turn establishes the inequality (3).

To prove (3) alternatively, we consider the function

$$f(x)=\Sigma a_j[\ln x_j-\ln(1-x_j)] \tag{4}$$

Since $f_j=a_j[\frac{1}{x_j}+\frac{1}{1-x_j}]$, and

$$f_{jk}=\begin{cases}a_j[-\frac{1}{x_j^2}+\frac{1}{(1-x_j)^2}], & k=j,\\ 0, & k\neq j,\end{cases}$$

$$Q=-\Sigma(1-2x_j)[\frac{u_j}{x_j(1-x_j)}]^2 \ . \tag{5}$$

From (5), if follows that $Q<0$ on I^n. Thus f is concave on I^n by Theorem 2 and the support of f at $x=x^*=(A,\cdots,A)$ (where $A=\langle a,x\rangle$) is

$$A(x)=\nabla f(x^*)(x-x^*)+f(x^*)=(\frac{1}{A}+\frac{1}{1-A})\langle a,x-x^*\rangle+f(x^*)$$

$$=(\frac{1}{A}+\frac{1}{1-A})(\langle a,x\rangle-A)+f(x^*)=f(x^*)=\ln A-\ln(1-A). \tag{6}$$

Hence the inequality (3) is established by a use of Theorem 3 on (4) and (6). Also, the sign of equality in (3) holds iff $x_1=\cdots=x_n$.

5. Continuous version

We state and prove the continuous version of a discrete Ky Fan inequality (3) of the complementary **A-G** type as follows:

Theorem 6 Let $\mathbf{I}$ denote the constant function on Ω whose value is 1 and $A=\langle w/\int wd\mu, f\rangle$ for w, $f\in U$. Then we have

$$\langle w/\int wd\mu, \ln f-\ln(1-f)\rangle\leqslant \ln A-\ln(1-A), \tag{7}$$

with equality iff $f=k$, where $k\in I$.

To prove Theorem 6, we consider the functional

$$K(f)=\langle w/\int wd\mu, \ln f-\ln(1-f)\rangle. \tag{8}$$

Since

$$\partial K=\langle w/\int wd\mu, f^{-1}+(1-f)^{-1}\rangle \text{ and}$$

$$\partial^2 K=\langle w/\int wd\mu, -f^{-2}+(1-f)^{-2}\rangle,$$

$$Q=-\langle w/\int wd\mu, (1-2f)[f(1-f)]^{-2}h^2\rangle. \tag{9}$$

From (9) if follows that $Q<0$ on U. Thus f is concave on U by Theorem 4 and the support of f at $f = A$ is

$$A(f) = \partial K(A)(f - A) + K(A) = \langle w/\int w d\partial, [A^{-1} + (1 - A)^{-1}](f - A) \rangle + K(A) = K(A) = \ln A - \ln(1 - A) \quad (10)$$

Hence the inequality (7) is established by the use of Theorem 5 on (8) and (10). The conclusion of the Theorem is now clear.

6. Determinantal anologue

In this section, we consider (cf. Beckenbach [3, p.57], Marcus and Minc [10, p.69], Minsky [12, p.394], and Wang [19, p.202], [20, p.109])

$$\Delta = \{ M \in \mathbf{M} | \langle Mz, z \rangle \langle \frac{1}{2} \langle z, z \rangle \},$$

where $\mathbf{M}$ = the set of positive definite matrices of order m, $z \in \mathbf{R}^m$. We use a sequence $M = (M_1, \cdots, M_n)$ of positive definite matrices $M_1, \cdots, M_n$ in Δ in place of the sequence $x = (x_1, \cdots, x_n)$ of positive numbers $x_1, \cdots, x_n$ in I considered above. For simplicity, we use 1 and $|W|$ to denote the identiry matrix of order m and the determinant of a matrix W respectively. We now state the determinantal anologue of a Ky Fan inequality (3) as follows:

Theorem 7. For $M_j \in \Delta$, $a_j > 0$ with $\Sigma a_j = 1, j = 1, \cdots, n$, we have

$$\Sigma a_j [\ln |M_j| - \ln |1 - M_j|] \leqslant \ln |\Sigma a_j M_j| - \ln |\Sigma a_j (1 - M_j)|, \quad (11)$$

with equality iff all M_j are equal.

Choosing M_j to be diagonal matrices $(\lambda_t^{(j)})_{t=1}^m$, $j = 1, \cdots, n$, of Δ, (11) becomes

$$\Sigma a_j [\ln \Pi_{t=1}^m \lambda_t^{(j)} - \ln \Pi_{t=1}^m (1 - \lambda_t^{(j)})]$$
$$\leqslant \ln \Pi_{t=1}^m (\Sigma a_j \lambda_t^{(j)}) - \ln \Pi_{t=1}^m [\Sigma a_j (1 - \lambda_t^{(j)})]. \quad (12)$$

A simple rearrangement of (12) yields

$$\Sigma_{t=1}^m f(\lambda_t) \leqslant \Sigma_{t=1}^m A(\lambda_t), \quad (13)$$

where $f(\lambda_t)$ and $A(\lambda_t)$ are given in (4) and (6) respectively. For each t, $f(\lambda_t) \leqslant A(\lambda_t)$ is a copy of inequality (3). Thus (13) holds and the case of equality is hereditary. This establishes Theorem 7 for the case of diagonal matrices of Δ.

However, by a use of Theorem 10.6.6 given on page 320 of Mirsky [12], Theorem 7 can be readily established for any n matrices M_j of Δ which cmmute in pairs. Practically, this is slightly better than the case of diagonal matrices. For general matrices of Δ, Theorem 7 requires further investigation (see below).

7. Remarks and a conjecture

The condition given in Theorem 6 for the equality case is stated as "$f = k$". In fact, it can be replaced by "$f = k$ almost everywhere" (e.g. see Halmos [6, p.86]). Now let us point out that in turn inequality (3) follows from Theorem 6 immediately by letting $f(x) = \Sigma x_j \chi_{[j, j+1]}$, $w(x) = \Sigma a_j \chi_{[j, j+1]}$ (see Halmos [6, p.84]), $d\mu =$

dx, $x \in \mathrm{R}$, and $\Omega = [1, n+1]$.

Instead of a direct investigation of Theorem 7 for the general case, we present the following.

Conjecture Let $f(M) = \ln|M| - \ln|1-M|$ be a real function defined on Δ. Then f is concave on Δ.

Since the Jensen concavity is equivalent to the usual concavity (cf. Hardy, Littlewood and Pólya [7, p. 71], or Mitrinović [12, p. 12]), the conjecture amounts to establishing the inequality

$$|M_1 M_2|^{1/2} / |(1-M_1)(1-M_2)|^{1/2} \leqslant |\frac{1}{2}(M_1+M_2)| / |1-\frac{1}{2}(M_1+M_2)|$$

for $M_1, M_2 \in \Delta$.

Finally, we conclude that it may be interesting and worthwhile to extend the inequality (3) to the case of M-matrices in the sense of Ando [1], [2].

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