J. Math. Res. & Exposition, 1988, No.4

A Free Boundary Problem for ODE Arising from Degenerate Parabolic Equations*

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Abstract Assume $B\in C[0,+\infty)\cap C^1(0,+\infty)$ and $B(v)$ is strictly increasing and concave. That the free boundary Problem for ODE

$$v''=-\frac{1}{2}\xi[B(v)]' \text{ for } \xi>\xi_0, \quad v(\xi_0)=0,$$
$$v'(\xi_0)=-\frac{a}{2}\xi_0, \quad \lim_{\xi\to+\infty}v(\xi)=\beta,$$

has a unique solution $(v(\xi),\xi_0)$ with $\xi_0\in[-(\frac{2\beta}{a})^{1/2},0)$ is proved. This problem arises, for example, from the investigation of the structure of discontinuous solutions for degenerate parabolic equations like $\frac{\partial u}{\partial t}=\frac{\partial^2 A(u)}{\partial x^2}$.

I. Introduction

Vol'pert A.I. and Hudjaev S.I. [1] indicated that a solution of the Cauchy problem for degenerate parabolic equations of the form

$$(1.1)\quad \begin{cases} \frac{\partial u}{\partial t}=\frac{\partial^2 A(u)}{\partial x^2} & \text{for } x\in R \text{ and } t>0, \\ u|_{t=0}=u_0(x) & \text{for } x\in R \end{cases}$$

with $A'(u)=a(u)\geq 0$ may be discontinuous somewhere. Wu Zhuoqun [2] showed that under the assumptions that

$$\begin{cases} a(u)=0 \text{ for } u<0 \text{ and } a(u)>0 \text{ for } u>0, \\ u_0(x)<0 \text{ for } x<0 \text{ and } u_0(x)>0 \text{ for } x>0, \end{cases}$$

the problem (1.1) can be reduced to the following free boundary problem

$$(1.2)\quad \begin{cases} \frac{\partial u}{\partial t}=\frac{\partial^2 A(u)}{\partial x^2} \text{ for } x>\lambda(t), \\ u|_{x=\lambda(t)}=0, \\ \frac{\partial A(u)}{\partial x}\Big|_{x=\lambda(t)}=u_0(\lambda(t))\lambda'(t), \\ u|_{t=0}=u_0(x) \text{ for } x>0 \end{cases}$$

where $x=\lambda(t)$, with $\lambda(0)=0$ and $\lambda'(t)<0$, is the line of discontinuity of a

* Received May 3, 1986.

solution u to (1.1). If, in addition,

$$u_0(x) = -\alpha \text{ for } x<0 \text{ and } u_0(x) = \beta \text{ for } x>0,$$

with positive constants α and β, then, seeking similarity solutions of (1.2) transforms it into the free boundary problem for ODE:

$$(1.3)\qquad \begin{cases} v'' = -\frac{1}{2}\xi[B(v)]' \text{ for } \xi > \xi_0 \\ v(\xi_0) = 0, \\ v'(\xi_0) = -\frac{\alpha}{2}\xi_0, \\ \lim\limits_{\xi \to +\infty} v(\xi) = A(\beta), \end{cases}$$

where $\xi = x/\sqrt{T}$, $v(\xi) = A(u(\xi))$ and $B(v)$ is the inverse function of $A(u)$ for $u>0$. In [2], Wu concentrated on the special case

$$A(u) = 0 \text{ for } u<0 \text{ and } A(u) = u^m \text{ for } u>0$$

with $m>1$, and proved the existence and uniquenss of solutions to (1.3).

What we want to do in this paper is to generalize the above result to the case of more general $A(u)$. Precisely, we assume that $A \in C(-\infty, +\infty)$, $A(u) = 0$ when $u \leqslant 0$ and $A(u)$ is continuously differentiable, strictly increasing and convex for $u>0$. Obviously, this implies that B, as the inverse of A, satisfies the condition:

$$(1.4)\qquad B \in C[0, +\infty) \cap C^1(0, +\infty) \text{ with } B(0) = 0 \text{ and } B(v) \text{ is strictly increasing and concave.}$$

In section 3 it is under the condition (1.4) thet we prove the existence and uniqueness result for (1.3). As a preliminary, Cauchy problems related to (1.3) are studied in greater detail in section 2.

Example

$$A(u) = \begin{cases} 0 \text{ for } u \leqslant 0, \\ u^m \text{ for } u>0, \end{cases} \quad \text{and} \quad A(u) = \begin{cases} 0 \text{ for } u \leqslant 0, \\ u^m \ln(1+u) \text{ for } u>0, \end{cases}$$

with $m>1$ setisfy the assumption stated above.

II. Relevant Cauchy problems

This section is devoted to the Cauchy problems of the form

$$(2.1)\qquad \begin{cases} v'' = -\frac{1}{2}\xi[B(v)]' \text{ for } \xi > \xi_0, \\ v(\xi_0) = a, \quad v'(\xi_0) = b \end{cases}$$

where ξ_0, a, b are constants with $\xi_0 > 0$ and $a, b \geqslant 0$.

Theorem 2.1 Assume that $B \in C[a, +\infty) \cap C^1(a, +\infty)$ and $B(v)$ is an increasing and concave function. Then (2.1) has a unique solution $v \in C^1[\xi_0, +\infty) \cap C^2(\xi_0, +\infty)$.

Proof It is easy to see that to seck a solution to (2.1) equals to find a continuous function v satisfying the integral equation

$$v(\xi)=a+b(\xi-\xi_0)+\frac{1}{2}\int_{\xi_0}^{\xi}(\xi-2\eta)B(v(\eta))d\eta \text{ for } \xi\geq\xi_0, \tag{2.2}$$

or equivalently,

$$v(\xi)=a+b\int_{\xi_0}^{\xi}\exp[-\frac{1}{2}\int_{\xi_0}^{\eta}\zeta B'(v(\zeta))d\zeta]d\eta \text{ for } \xi\geq\xi_0. \tag{2.3}$$

Define

$$\Omega=\{v\in C[\xi_0,0]: a+b(\xi-\xi_0)\leq v(\xi)\leq a+b'(\xi-\xi_0)\},$$

where

$$b'=b\exp[\frac{|\xi_0|}{2b}(B(a+b|\xi_0|)-B(a))]. \tag{2.4}$$

It is clear that Ω is a closed convex set in $C[\xi_0,0]$. Introduce an operator T: $\Omega\to C[\xi_0,0]$ by assigning

$$w(\xi)\equiv Tv(\xi)\equiv a+b\int_{\xi_0}^{\xi}\exp[-\frac{1}{2}\int_{\xi_0}^{\eta}\zeta B'(v(\zeta))d\zeta]d\eta \text{ for } \xi_0\leq\xi\leq 0.$$

Since B' is nonnegative and nonincreasing, we have

$$b\leq w'(\xi)=b\exp[-\frac{1}{2}\int_{\xi_0}^{\xi}\zeta B'(v(\zeta))d\zeta]\leq b'$$

and hence

$$a\leq a+b(\xi-\xi_0)\leq w(\xi)\leq a+b'(\xi-\xi_0)\leq a+b'|\xi_0|.$$

for any $v\in\Omega$ and any $\xi\in[\xi_0,0]$. This means that $T(\Omega)\subset\Omega$ and that, by the Alzera lemma, $T(\Omega)$ is a precompact set in $C[\xi_0,0]$. In addition, $B\in C[a,+\infty)\cap C^1(a,+\infty)$ ensures the continuity of T. Thus, Schauder's fixed point theorem gives that the integral equation (2.3) has at least one solution defined on the interval $[\xi_0,0]$. It is easy to see that this solution can be continued to $[\xi_0,+\infty)$. The existence of solutions to (2.1) is thus proved.

To show the uniqueness of solutions, let $v_1(\xi)$ and $v_2(\xi)$ be two solutions of (2.1), and write

$$\bar{\xi}=\sup\{\xi\geq\xi_0: v_1(\eta)=v_2(\eta) \text{ for } \xi_0\leq\eta\leq\xi\}.$$

It suffices to confirm $\bar{\xi}=+\infty$. If $\bar{\xi}<+\infty$, $v_i(\xi)$, $i=1,2$, could have the following expressions:

$$v_i(\xi)=v_i(\bar{\xi})+v_i'(\bar{\xi})(\xi-\bar{\xi})+\frac{1}{2}\int_{\bar{\xi}}^{\xi}(\xi-2\eta)B(v_i(\eta))d\eta \text{ for } \xi\geq\bar{\xi}, \tag{2.5}$$

and

$$v_i(\xi)=v_i(\bar{\xi})+v_i'(\bar{\xi})\int_{\bar{\xi}}^{\xi}\exp[-\frac{1}{2}\int_{\bar{\xi}}^{\eta}\zeta B'(v_i(\zeta))d\zeta]d\eta \text{ for } \xi\geq\bar{\xi}. \tag{2.6}$$

Set $v(\xi)=v_1(\xi)-v_2(\xi)$. From (2.5) we get

$$v(\xi)=\frac{1}{2}\int_{\bar{\xi}}^{\xi}(\xi-2\eta)v(\eta)d\eta\int_0^1 B'(tv_1(\eta)+(1-t)v_2(\eta))dt \text{ for } \xi\geq\bar{\xi} \tag{2.7}$$

since $v_1(\bar{\xi}) = v_2(\bar{\xi})$, $v_1'(\bar{\xi}) = v_2'(\bar{\xi})$. From (2.6) and

$$v_i'(\bar{\xi}) = b\exp[-\frac{1}{2}\int_{\xi_0}^{\bar{\xi}} \zeta B'(v_i(\zeta))d\zeta] > 0,$$

we see that

$$v_i(\xi) \geqslant v_i(\bar{\xi}) \quad \text{for } \xi \geqslant \bar{\xi}.$$

By (2.3)

$$v_i(\xi) \geqslant a + b(\xi - \xi_0) \quad \text{for } \xi_0 \leqslant \xi \leqslant 0.$$

Hence (2.7) implies that, for any $\delta > 0$ small and for any $\xi \in [\bar{\xi}, \bar{\xi} + \delta]$,

$$|v(\xi)| \leqslant \frac{1}{2}\int_{\bar{\xi}}^{\xi} |\xi - 2\eta| d\eta B'(v_1(\bar{\xi})) \max_{\bar{\xi} \leq \zeta \leq \bar{\xi}+\delta} |v(\zeta)|$$

$$\leqslant \frac{1}{2} B'(v_1(\bar{\xi}))(|\bar{\xi}| + \delta)\delta \max_{\bar{\xi} \leq \zeta \leq \bar{\xi}+\delta} |v(\zeta)| \quad \text{if } \bar{\xi} > \xi_0$$

or

$$|v(\xi)| \leqslant \frac{1}{2}(|\xi_0| + \delta)\int_{\xi_0}^{\xi} B'(a + b(\eta - \xi_0))d\eta \max_{\xi_0 \leq \zeta \leq \xi_0+\delta} |v(\zeta)|$$

$$\leqslant \frac{1}{2b}(|\xi_0| + \delta)[B(a + b\delta) - B(a)] \max_{\xi_0 \leq \zeta \leq \xi_0+\delta} |v(\zeta)| \quad \text{if } \bar{\xi} = \xi_0.$$

This is impossible for δ small enough. Thus $\bar{\xi} = +\infty$, i.e., $v_1(\xi) = v_2(\xi)$ for all $\xi \geqslant \xi_0$.

Now we cliscuss some properties of solutions to (2.1), which will be used in section 3. First we have

Theorem 2.2 Let $v \in C^1[\xi_0, +\infty) \cap C^2(\xi_0, +\infty)$ be a solution of (2.1) with $\xi_0 < 0$, $a \geqslant 0, b > 0$ and $B'(v) \geqslant 0$. Then v is an strictly increasing function with v' increasing on $[\xi_0, 0]$ and decreasing on $[0, +\infty)$. In addition,

(2.8) $$a \leqslant v(\xi) \leqslant a + b'(\xi - \xi_0) \quad \text{for } \xi \geqslant \xi_0$$

where b' is given in (2.4).

Proof From (2.3) we see that

$$v(\xi) = b\exp[-\frac{1}{2}\int_{\xi_0}^{\xi} \eta B'(v(\eta))d\eta] > 0 \quad \text{for } \xi \geqslant \xi_0$$

and from (2.1) we obtain $\xi v''(\xi) \leqslant 0$ for $\xi > \xi_0$. This implies the first part of the theorem and $v(\xi) \geqslant a$.

For $\xi_0 \leqslant \xi \leqslant 0$, from (2.3) we get

$$v(\xi) \leqslant a + b'(\xi - \xi_0).$$

For $\xi > 0$,

$$v(\xi) = a + b(\int_{\xi_0}^{0} + \int_0^{\xi})\exp[-\frac{1}{2}\int_{\xi_0}^{\eta} \zeta B'(v(\zeta))d\zeta]d\eta$$

$$\leqslant a + b'(-\xi_0) + b\int_0^{\xi}\exp[-\frac{1}{2}(\int_{\xi_0}^{0} + \int_0^{\eta})\zeta B'(v(\zeta))d\zeta]d\eta$$

$$\leqslant a - b'\xi_0 + b'\int_0^{\xi}\exp[-\frac{1}{2}\int_0^{\eta}\zeta B'(v(\zeta))d\zeta]d\eta < a + b'(\xi - \xi_0).$$

Hence (2.8) is true.

From now on we denote by $v(\xi;\xi_0,a,b)$ the solution of the problem (2.1).

Theorem 2.3 Assume thet $B\in C[0,+\infty)\cap C^1(0,+\infty)$ and $B(v)$ is an increasing and concave function. Then $v(\xi;\xi_0,a,b)$ depends continuously on $\xi_0<0$, $a\geqslant 0$ and $b>0$.

Proof Fixing $\xi_0<0, a\geqslant 0$ and $b>0$, we want to prove

$$v(\xi;\bar{\xi},\bar{a},\bar{b})\to v(\xi;\xi_0,a,b) \quad \text{for } \xi>\xi_0$$

when $\bar{\xi}_0<0,\bar{a}\geqslant 0,\bar{b}>0$, and $\bar{\xi}_0\to\xi_0$, $\bar{a}\to a$, $\bar{b}\to b$. It suffices to show that for any $\xi_1>0$, any $\delta>0$ small and for $\xi_0+\delta\leqslant\xi\leqslant\xi_1$

$$(2.9)\qquad v(\xi;\xi_{0n},a_n,b_n)\to v(\xi;\xi_0,a,b)$$

when $|\xi_{0n}-\xi_0|<\delta$, $\xi_{0n}<0, a_n\geqslant 0, b_n>0$ and $\xi_{0n}\to\xi_0$, $a_n\to a$, $b_n\to b$.

By (2.2) we have, for $\xi\geqslant\xi_{0n}$,

$$(2.10)\qquad v(\xi;\xi_{0n},a_n,b_n)=a_n+b_n(\xi-\xi_{0n})+\frac{1}{2}\int_{\xi_{0n}}^{\xi}(\xi-2\eta)B(v(\eta;\xi_{0n},a_n,b_n))d\eta$$

$$v'(\xi;\xi_{0n},a_n,b_n)=b_n-\frac{1}{2}\xi B(v(\xi;\xi_{0n},a_n,b_n))+\frac{1}{2}\int_{\xi_{0n}}^{\xi}B(v(\eta;\xi_{0n},a_n,b_n))d\eta.$$

Applying Theorem 2.2 to $v(\xi;\xi_{0n},a_n,b_n)$, we get

$$0\leqslant a_n\leqslant v(\xi;\xi_{0n},a_n,b_n)\leqslant a_n+b_n'(\xi_1-\xi_{0n}),$$

and $$0\leqslant v'(\xi;\xi_{0n},a_n,b_n)\leqslant v'(0;\xi_{0n},a_n,b_n)\leqslant b_n+\frac{|\xi_{0n}|}{2}B(a_n+b_n'(\xi_1-\xi_{0n}))$$

where $b_n'=b_n\exp[\frac{|\xi_{0n}|}{2b_n}(B(a_n+b_n|\xi_{0n}|)-B(a_n))]$. It is clear that for n large enough $v(\xi;\xi_{0n},a_n,b_n)$ and $v'(\xi;\xi_{0n},a_n,b_n)$ are uniformly bounded on $[\xi_0+\delta,\xi_1]$. According to the Alzera lemma, there exists a subsequence $\{n_k\}$ such that $v(\xi;\xi_{0n_k},a_{n_k},b_{n_k})$ uniformly converges to some continuous function $w(\xi)$. Letting $k\to+\infty$ in (2.10) with n_k in place of n we see that

$$w(\xi)=a+b(\xi-\xi_0)+\frac{1}{2}\int_{\xi_0}^{\xi}(\xi-2\eta)B(w(\eta))d\eta,$$

which means that $w(\xi)$ satisfies (2.1). By uniqueness and its proof, $w(\xi)=v(\xi;\xi_0,a,b)$ and (2.9) holds.

Theorem 2.4 Under the assumption of Theorem 2.3, if $\xi_{10}\leqslant\xi_{20}<0$, $a_1\geqslant a_2\geqslant 0$, $b_1\geqslant b_2>0$, and $|\xi_{10}|+a_1+b_1\geqslant|\xi_{20}|+a_2+b_2$,

then

$$(2.11)\qquad v(\xi;\xi_{10},a_1,b_1)>v(\xi;\xi_{20},a_2,b_2) \text{ for } \xi\geqslant\xi_{02}.$$

Proof Let us first consider the case

$$(2.12)\qquad \xi_{10}=\xi_{20}=\xi_0<0,\quad a_1>a_2\geqslant 0, b_1=b_2=b>0$$

and write

$$v(\xi;\xi_0,a_i,b)=v_i(\xi),\quad i=1,2.$$

From $v_1(\xi_0)=a_1>a_2=v_2(\xi_0)$ and the continuity of $v_i(\xi)$ it follows that

$$v_1(\xi) > v_2(\xi) \text{ for } \xi_0 \leqslant \xi < \xi_0+\delta \text{ and } \delta > 0 \text{ small.}$$

Suppose that (2.11) were not true, there would be $\bar{\xi} \in [\xi_0+\delta, +\infty)$ such thut

$$v_1(\xi) > v_2(\xi) \text{ for } \xi_0 \leqslant \xi < \bar{\xi} \text{ and } v_1(\bar{\xi}) = v_2(\bar{\xi}).$$

But, by the monotonity of B and B', we have

$$v_1(\bar{\xi}) = a_1 + b(\bar{\xi}-\xi_0) + \frac{1}{2}\int_{\xi_0}^{\bar{\xi}} (\bar{\xi}-2\eta) B(v_1(\eta)) \mathrm{d}\eta$$

$$> a_2 + b(\bar{\xi}-\xi_0) + \frac{1}{2}\int_{\xi_0}^{\bar{\xi}} (\bar{\xi}-2\eta) B(v_2(\eta)) \mathrm{d}\eta = v_2(\bar{\xi})$$

if $\bar{\xi} \leqslant 0$ and

$$v_1(\bar{\xi}) = v_1(0) + v_1'(0)\int_0^{\bar{\xi}} \exp\left[-\frac{1}{2}\int_0^{\eta} \zeta B'(v_1(\zeta))\mathrm{d}\zeta\right]\mathrm{d}\eta$$

$$> v_2(0) + v_2'(0)\int_0^{\bar{\xi}} \exp\left[-\frac{1}{2}\int_0^{\eta} \zeta B'(v_2(\zeta))\mathrm{d}\zeta\right]\mathrm{d}\eta = v_2(\bar{\xi})$$

if $\bar{\xi} > 0$ since $v_1(0) > v_2(0)$ and $v_1'(0) \geqslant v_2'(0) \geqslant 0$. This contradicts $v_1(\bar{\xi}) = v_2(\bar{\xi})$. Therefore (2.11) holds in the case of (2.12).

By using the saime argument we can prove (2.11) for the case

$$\xi_{10} = \xi_{20} < 0,\ a_1 = a_2 \geqslant 0,\ b_1 > b_2 > 0 .$$

Now we assnme

$$\xi_{10} < \xi_{20} < 0,\ a \geqslant 0,\ b > 0.$$

By (2.2),

$$v(\xi_{20}; \xi_{10}, a, b) = a + b(\xi_{20} - \xi_{10}) + \frac{1}{2}\int_{\xi_{10}}^{\xi_{20}} (\xi_{20}-2\eta) B(v(\eta; \xi_{10}, a, b))\mathrm{d}\eta \equiv a' > a,$$

$$v'(\xi_{20}; \xi_{10}, a, b) = b - \frac{1}{2}\xi_{20} B(v(\xi_{20}; \xi_{10}, a, b)) + \frac{1}{2}\int_{\xi_{10}}^{\xi_{20}} B(v(\eta; \xi_{10}, a, b))\mathrm{d}\eta \equiv b' > b.$$

In view of the uniqueness of solutions,

$$v(\xi; \xi_{10}, a, b) = v(\xi; \xi_{20}, a', b') \text{ for } \xi \geqslant \xi_{20} .$$

According to what have just been proved

$$v(\xi; \xi_{10}, a, b) = v(\xi; \xi_{20}, a', b') > v(\xi; \xi_{20}, a, b) \text{ for } \xi \geqslant \xi_{20} .$$

Ⅲ. Free boundary Problem

Let us back to the free boundary problem (1.3) and write β instead of $A(\beta)$ for simplicity, i.e., consider the problem

$$(3.1)\qquad \begin{cases} v'' = -\frac{1}{2}\xi[B(v)]' \text{ for } \xi > \xi_0, \\ v(\xi_0) = 0, \\ v'(\xi_0) = -\frac{a}{2}\xi_0, \\ \lim\limits_{\xi \to +\infty} v(\xi) = \beta . \end{cases}$$

Throughout this section $B(v)$ is assumed to satisfy the condition

(3.2) $B \in C[0, +\infty) \cap C^1(0, +\infty)$ is strictly increasing and concave. Applying the results of the last section to the cauchy problem

$$(3.3)\qquad \begin{cases} v'' = -\frac{1}{2}\xi[B(v)]' & \text{for } \xi > \xi_0, \\ v(\xi_0) = 0, \\ v'(\xi_0) = -\frac{a}{2}\xi_0, \end{cases}$$

we get

Theorem 3.1 Under the condition (3.2), (3.3) has a unique solution $v = v(\xi; \xi_0, a)$, which depends continuously on $\xi_0 < 0$ and $a > 0$, and strictly increases with ξ, $|\xi_0|$ and a.

If, in addition,

$$(3.4)\qquad \lim_{v\to+\infty} v^2 B'(v) = +\infty$$

then there exists a positive constant M snch that

$$(3.5)\qquad M > v(\xi; \xi_0, a) > \begin{cases} \frac{a}{2}|\xi_0|(\xi - \xi_0) & \text{for } \xi_0 < \xi < 0, \\ \frac{a}{2}\xi_0^2 & \text{for } \xi > 0. \end{cases}$$

Proof It remains to Show thet (3.5) is true if (3.2) and (3.4) hold.

From (2.3) with $a = 0$ and $b = \frac{a}{2}|\xi_0|$, we have

$$(3.6)\qquad v(\xi; \xi_0, a) = \frac{a}{2}|\xi_0|\int_{\xi_0}^{\xi} \exp[-\frac{1}{2}\int_{\xi_0}^{\eta} \zeta B'(v(\zeta; \xi_0, a))d\zeta]d\eta.$$

Set

$$\lambda = \frac{a}{2}|\xi_0|\exp[\frac{1}{a}(B(\frac{a}{2}\xi_0^2) - B(0))]$$

and choose $M > 0$ such that

$$\lambda(|\xi_0| + \sqrt{\frac{\pi}{B'(M)}}) < M.$$

This is possible because of (3.4). It is easily seen from (3.6) that

$$(3.7)\qquad \frac{a}{2}|\xi_0|(\xi - \xi_0) < v(\xi; \xi_0, a) < \lambda|\xi_0| < M \quad \text{for } \xi_0 < \xi < 0.$$

When $\xi > 0$ we have

$$(3.8)\qquad v(\xi; \xi_0, a) > v(0; \xi_0, a) > \frac{a}{2}\xi_0^2.$$

If there exists $\xi_1 > 0$ such that

$$v(\xi; \xi_0, a) = v(\xi) < M \text{ for } \xi_0 < \xi < \xi_1 \text{ and } v(\xi_1) = M,$$

then for $0 < \xi < \xi_1$

$$v(\xi) = \frac{a}{2}|\xi_0|(\int_{\xi_0}^{0} + \int_0^{\xi})\exp[-\frac{1}{2}\int_{\xi_0}^{\eta}\zeta B'(v(\zeta))d\zeta]d\eta$$

$$< \lambda|\xi_0| + \lambda\int_0^{\xi}\exp[-\frac{1}{4}B'(M)\eta^2]d\eta < \lambda(|\xi_0| + \sqrt{\frac{\pi}{B'(M)}}) < M,$$

which contradicts $v(\xi_1) = M$. This together with (3.7) and (3.8) gives (3.5).

By Theorem 3.1, under the conditions (3.2) and (3.4),

$$y(\xi_0, a) = \lim_{\xi \to +\infty} v(\xi)$$

is well defined on $\xi_0<0$ and $a>0$. We are going to find a $\xi_0<0$ such that $y(\xi_0, a)=\beta$. For thiz ξ_0, the solution of (3.3) also solves (3.1).

To seek such a $\xi_0<0$, the following properties of $y(\xi_0, a)$ are crucial.

Theorem 3.2 Under the conditions (3.2) and (3.4),

i) $\lim_{\xi \to +\infty} v(\xi; \xi_0, a) = y(\xi_0, a)$ uniformly on $-L \leqslant \xi_0 < 0$ and $0<a_1 \leqslant a \leqslant a_2$ for any positive L, a_1 and a_2;

ii) $y(\xi_0, a) \geqslant \frac{a}{2}\xi_0^2$ for any $\xi_0<0$ and $a>0$;

iii) $y(\xi_0, a)$ is continuous on $\xi_0<0$ and $a>0$;

iv) If $\xi_{10} \leqslant \xi_{20} < 0$, $a_1 \geqslant a_2 > 0$, $|\xi_{10}|+a_1 > |\xi_{20}|+a_2$, then $y(\xi_{10}, a_1) > y(\xi_{20}, a_2)$.

Proof It is easy to see form the proof of (3.5) that we can choose $M>0$ independent of $\xi_0 \in [-L, 0)$ and $a \in [a_1, a_2]$. Hence, for $\xi>0$, we have

$$0 < y(\xi_0, a) - v(\xi; \xi_0, a) = \frac{a}{2}|\xi_0| \int_\xi^\infty \exp[-\frac{1}{2}\int_{\xi_0}^\eta \zeta B'(v(\zeta; \xi_0, a))d\zeta] d\eta$$

$$\leqslant \frac{a}{2}|\xi_0| \exp[\frac{|\xi_0|}{2}\int_{\xi_0}^0 B'(\frac{a}{2}|\xi_0|(\zeta - \xi_0))d\zeta] \int_\xi^\infty \exp[-\frac{1}{4}B'(M)\eta^2] d\eta$$

$$\leqslant \frac{a|\xi_0|}{\sqrt{B'(M)}} \exp[\frac{1}{a}(B(\frac{a}{2}\xi_0^2) - B(0))] \int_\xi^\infty e^{-\eta^2} d\eta$$

$$\leqslant \frac{a_2 L}{\sqrt{B'(M)}} \exp[\frac{1}{a_1}(B(\frac{a_2}{2}L^2) - B(0))] \int_\xi^\infty e^{-\eta^2} d\eta$$

which proves i). ii) follows from (3.5). iii) is a consequence of Theorem 3.1 and i). As for iv) we have

$$y(\xi_{10}, a_1) = v(0; \xi_{10}, a_1) + v'(0; \xi_{10}, a_1) \int_0^\eta \exp[-\frac{1}{2}\int_0^\eta \zeta B'(v(\zeta; \xi_{10}, a_1))d\zeta] d\eta$$

$$> v(0; \xi_{20}, a_2) + v'(0; \xi_{20}, a_2) \int_0^\infty \exp[-\frac{1}{2}\int_0^\eta \zeta B'(v(\zeta; \xi_{20}, a_2))d\zeta] d\eta = y(\xi_{20}, a_2).$$

By Theorem 3.2, for fixed $a>0$, $y(\xi_0, a)$ is a strictly increasing and continuons function on $\xi_0<0$ and

$$y(-(\frac{2\beta}{a})^{1/2}, a) \geqslant \frac{a}{2}\,\frac{2\beta}{a} = \beta.$$

In addition,

$$\lim_{\xi_0 \to 0^+} y(\xi_0, a) = \lim_{\xi_0 \to 0^+} \frac{a|\xi_0|}{2}\int_{\xi_0}^\infty \exp[-\frac{1}{2}\int_{\xi_0}^\eta \zeta B'(v(\zeta; \xi_0, a))d\zeta] d\eta = 0.$$

Hence, there exists a unique $\xi_0 \in [-(\frac{2\beta}{a})^{1/2}, 0)$ such that

$$y(\xi_0, a) = \beta.$$

In other words, $(v(\xi; \xi_0, a), \xi_0)$ is the unique solution of (3.1). If we change notatieons and denote by $(v(\xi;\alpha,\beta),\ \xi_0(\alpha,\beta))$ thc solution of (3.1), the results that have just been obtained can be stated as

Theorem 3.3 Assume (3.2). For $\alpha,\beta>0$, (3.1) has a unique solution $(v(\xi;\alpha,\beta),\ \xi_0(\alpha,\beta))$ with $\xi_0(\alpha,\beta)\in[-(\frac{2\beta}{\alpha})^{1/2},0)$ and $v\in C^1[\xi_0,+\infty)\cap C^2(\xi_0,+\infty)$. Both $v(\xi;\alpha,\beta)$ and $\xi_0(\alpha,\beta)$ are continuous in α and β. $v(\xi;\alpha,\beta)$ strictly increases in ξ and β. and $\xi_0(\alpha,\beta)$ strictly in α and $-\beta$.

Proof Under the additional restriction (3.4) it remains to verify the monotonity of $v(\xi;\alpha,\beta)$ and $\xi_0(\alpha,\beta)$. By Theorem 3.2, for fixed $\alpha>0$, $\xi_0(\alpha,\beta)$ as the implicit function defined by $y(\xi_0,\alpha)=\beta$ is strictly decreasing in β, and for fixed $\beta>0$ it is strictly increasing in α. And hence $v(\xi;\alpha,\beta)$ has the claimed monotonity by Theorem 3.1.

For general $B(v)$ just satisfying (3.2), define

$$\overline{B}(v)=\begin{cases}B(v) & \text{for } 0\leqslant v\leqslant\beta,\\ B(\beta)+B'(\beta)(v-\beta) & \text{for } v>\beta.\end{cases}$$

As has been proved, (3.1) with $B(v)$ replaced by $\overline{B}(v)$ has a unique solution $(v(\xi),\xi_0)$. Obviously, $0\leqslant v(\xi)\leqslant\beta$. Therefore, $(v(\xi),\xi_0)$ is also the unique solution of (3.1) itself.

Acknowledgement The author wishes to thank the Centre for Mathematical Analysis, the Australian National University. This work was done while he was visiting there.

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