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Propositional Calculus System of Medium Logic (Ⅲ)*

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The papar is a continuation of [1], [2], [4], [5]. In this papar we shall introduce conjunction symbol "$\wedge$", disjunction symbol "$\vee$" and equality value symbol "$\leftrightarrow$" as defining symbols of MP, which are read as "and", "or", "if and only if" respectively. With these symbols we shall go on to prove the formal theorems of MP. The list of the following formal theorems are in succession of refernce [5].

New proposition connectives symbols $\wedge$、$\vee$ and $\leftrightarrow$, of course, may be added to the vocabulary of MP. and we can develop a new propositional calculus system on the basis of the vocabulary and rules of formation and inference of the extended MP. Then we discuss its relation with MP. However, all these works will be left in future.

Definitjon D($\vee$) $A \vee B =_{df}$ ╕ D($\wedge$) $A \wedge B =_{df}$ ╕$(A \rightarrow$ ╕$B)$,

D($\leftrightarrow$) $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A) =_{df}$ ╕[$(A \rightarrow B) \rightarrow$ ╕$(B \rightarrow A)$]

Theorem 23 MP:

[1] $A \wedge B \vdash A,\ B$

[2] $A,\ B \vdash A \wedge B$

[3] ╕$A \vdash$ ╕$(A \wedge B)$

[4] ╕$B \vdash$ ╕$(A \wedge B)$

[5] $\sim A,\ B \vdash \sim (A \wedge B)$

[6] $\sim A,\ \sim B \vdash \sim (A \wedge B)$

[7] Let / , * be two different symbols among ╕、~、⊙, Where ⊙ denots empty, then $\vdash$ ┐(/$A \wedge *A$). This is the law of consistent in MP.

Proof of [1]:

(1) $A \wedge B$

(2) ╕$(A \rightarrow$ ╕$B)$ (1)(D($\wedge$))

(3) A (2)(╕.)

(4) ╕╕B (2)(╕.)

(5) B (4)(MP3[1])

Proof of [2]: Using D($\wedge$)、(╕╕.) and (╕.).

Proof of [3]: Using D($\wedge$)、(╕╕.) and Theorem 5[1].

Proof of [4]: Using D($\wedge$)、(╕╕.) and Theorem 1[2].

Proof of [5]:

(1) $\sim A$

(2) [illegible] B

(3) [illegible] $A \wedge B$

(4) [illegible] A (3)(MP23[1])

(5) [illegible] A (1)(Y)

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(6) □□□ ㅋ(∧ *B*)
(7) □□□ ㅋㅋ(*A*→ ㅋ*B*)　　(6)(D(∧))
(8) □□□ *A*→ ㅋ*B*　　(7)(MP3[1])
(9) □□□ ㅋ*B*　　(1)(8)(→_)
(10) □□□ ¬*B*　　(9)(Y ㅋ)
(11) □~(*A*∧ *B*)　　(4)(5)(2)(10)(MP6[2])

Proof of [6]:

(1) ~ *A*
(2) □~ *B*
(3) □□□□□□ *A*∧ *B*
(4) □□□□□□ *A*　　(3)(MP23[1])
(5) □□□□□□¬ *A*　　(1)(Y~)
(6) □□□ ㅋ(*A*∧ *B*)
(7) □□□ ㅋㅋ(*A*→ ㅋ*B*)　　(6)(D(∧))
(8) □□□ *A*→ ㅋ*B*　　(7)(MP3[1])
(9) □□□ ㅋ*B*　　(1)(8)(→_)
(10) □□□¬~ *B*　　(9)(Y ㅋ)
(11) □~(*A*∧ *B*)　　(4)(5)(2)(10)(MP6[2])

Proof of [7]: First we prove ⊢¬ (ㅋ*A*∧~ *A*)

(1) □□ ㅋ*A*∧~ *A*
(2) □□ ㅋ*A*　　(1)(MP23[1])
(3) □□~ *A*　　(1)(MP23[1])
(4) □□¬ ~ *A*　　(2)(Y ㅋ)
(5) ¬ (ㅋ*A*∧~ *A*)　　(3)(4)(¬+)

Next, we prove ⊢¬ (*A*∧~ *A*)

(1) □□ *A*∧~ *A*
(2) □□ *A*　　(1)(MP23[1])
(3) □□~ *A*　　(1)(MP23[1])
(4) □□¬ *A*　　(3)(Y~)
(5) ¬ (*A*∧~ *A*)　　(2)(4)(¬+)

Now we prove ⊢¬ (*A*∧ ㅋ*A*)

(1) □□ *A*∧ ㅋ*A*
(2) □□ *A*　　(1)(MP23[1])
(3) □□ ㅋ*A*　　(1)(MP23[1])
(4) □□¬ *A*　　(3)(Y ㅋ)
(5) ¬ (*A*∧ ㅋ*A*)　　(2)(4)(¬+)

Theorem 24 MP:

[1] If *A*⊢*C* and *B*⊢*C* , then *A*∨*B*⊢*C* .
[2] *A*⊢*A*∨*B*, *B*∨*A*
[3] ㅋ(*A*∨*B*)⊢ ㅋ*A*, ㅋ*B*
[4] ㅋ*A*, ㅋ*B*⊢ ㅋ(*A*∨*B*)
[5] ~*A*, ㅋ*B*⊢~(*A*∨*B*)
[6] ~*A*, ~*B*⊢~(*A*∨*B*)
[7] ⊢*A*∨~*A*∨ ㅋ*A*

Proof of [1]:

(1)	$A \vee B$	
(2)	□□□□□□□ A	
(3)	□□□□□□□ C	(2) hypothesis
(4)	□□□□□ ㅋA	
(5)	□□□□□ ㅋ$A \to B$	(1)(D($\vee$))
(6)	□□□□□ B	(4)(5)($\to_-$)
(7)	□□□□□ C	(6) hypothesis
(8)	□□□ ~A	
(9)	□□□ ㅋ$A \to B$	(1)(D($\vee$))
(10)	□□□ ~ㅋA	(8)(MP 4[2])
(11)	□□□ B	(10)(9)($\to_-$)
(12)	□□□ C	(11) hypothesis
(13)	C	(3)(7)(12)(MP 11[5])

Proof of [2]: "$A \vdash A \vee B$" follows from D($\vee$)、(ㅋㅋ$_+$) and Theorem 5[1]. By D($\vee$) and Theorem 1[2] we have $A \vdash B \vee A$.

Proof of [3]: First we prove ㅋ$(A \vee B) \vdash$ ㅋA

(1)	ㅋ$(A \vee B)$	
(2)	□□□□□ ~A	
(3)	□□□□□□□□ B	
(4)	□□□□□□□□ $A \vee B$	(3)(MP 24[2])
(5)	□□□□□□□□ ¬ ㅋ$(A \vee B)$	(4)(Y)
(6)	□□□□□□□ ~B	
(7)	□□□□□□□ ~ ㅋA	(2)(MP 4[2])
(8)	□□□□□□□ ~(ㅋ$A \to B$)	(6)(7)(MP 12[1])
(9)	□□□□□□□ ~$(A \vee B)$	(8)(D($\vee$))
(10)	□□□□□□□ ¬ ㅋ$(A \vee B)$	(9)(Y$_\sim$)
(11)	□□□□□ ㅋB	(1)(5)(1)(10)(MP 6[3])
(12)	□□□□□ ~ㅋ A	(2)(MP 4[2])
(13)	□□□□□ ~(ㅋ$A \to B$)	(11)(12)(MP 12[2])
(14)	□□□□□ ~$(A \vee B)$	(13)(D($\vee$))
(15)	□□□□□ ¬ ㅋ$(A \vee)$	(14)(Y$_\sim$)
(16)	□□□ A	
(17)	□□□ $A \vee B$	(16)(MP 24[2])
(18)	□□□ ¬$(A \vee B)$	(1)(Y$_ㅋ$)
(19)	ㅋA	(1)(15)(17)(18)(MP 6[3])

Similarly, we can get ㅋ$(A \vee B) \vdash$ ㅋB.

Proof of [4]:

(1) ∃A
(2) □ ∃B
(3) □□□□□ A ∨ B
(4) □□□□□ ∃A → B　　(3)(D(∨))
(5) □□□□□ B　　(1)(4)(→_)
(6) □□□□□¬ B　　(2)(Y∃)
(7) □□□ ~(A ∨ B)
(8) □□□ ~(∃A → B)　　(7)(D(∨))
(9) □□□ ~B　　(1)(8)(MP 12[4])
(10) □□□¬ ∃B　　(9)(Y~)
(11) □ ∃(A ∨ B)　　(5)(6)(2)(10)(MP 6[3])

Proof of [5]:

(1 — ~A
(2) □ ~B
(3) □□□□□ A ∨ B
(4) □□□□□ ∃A → B　　(3)(D(∨))
(5) □□□□ ~∃A　　(1)(MP 4[2])
(6) □□□□□ B　　(5)(4)(→_)
(7) □□□□□¬ B　　(2)(Y~)
(8) □□□ ∃(A ∨ B)
(9) □□□ ∃A　　(8)(MP 24[3])
(10) □□□¬ ~A　　(9)(Y∃)
(11) □ ~(A ∨ B)　　(6)(7)(1)(10)(MP 6[2])

Proof of [6]: Similar to [5].

Proof of [7]:

(1) □□□□□ A
(2) □□□□□□ A ∨ ~A　　(1)(MP 24[2])
(3) □□□□□□ A ∨ ~A ∨ ∃A　　(2)(MP 24[2])
(4) □□□□ ~A
(5) □□□□ A ∨ ~A　　(4)(MP 24[2])
(6) □□□□□ A ∨ ~A ∨ ∃A　　(5)(MP 24[2])
(7) □□□ ∃A
(8) □□ ~A ∨ ∃A　　(7)(MP 24[2])
(9) □□ A ∨ ~A ∨ ∃A　　(8)(MP 24[2])
(10) A ∨ ~A ∨ ∃A　　(3)(6)(9)(MP 11[5])

Theorem 25 MP:

[1] ∃(A ∨ B) ⊢⊣ ∃A ∧ ∃B　　[2] ∃(A ∧ B) ⊢⊣ ∃A ∨ ∃B
[3] ∃(A ∨ B) ⊨⫤ ∃A ∨ ∃B　　[4] ∃(A ∨ B) ⊨⫤ ∃A ∨ ∃B

Proof of [1]: By Theorems 23[1]、[2] and Theorems 24[3]、[4].

Proof of [2]: First we prove $\neg(A\wedge B)\vdash\neg A\vee\neg B$

(1) $\neg(A\wedge B)$

(2) $\neg\neg(A\rightarrow\neg B)$ (1)(D(∧))

(3) $A\rightarrow\neg B$ (2)(MP3(1))

(4) $\neg\neg A\rightarrow\neg B$ (3)(MP18[1])

(5) $\neg A\vee\neg B$ (4)(D(∨))

Now we prove $\neg A\vee\neg B\vdash\neg(A\wedge B)$

(1) $\neg A\vee\neg B$

(2) $\neg\neg A\rightarrow\neg B$ (1)(D(∨))

(3) $A\rightarrow\neg B$ (2)(MP18[1])

(4) $\neg\neg(A\rightarrow\neg B)$ (3)(ㄱㄱ+)

(5) $\neg(A\wedge B)$ (4)(D(∧))

Proof of [3]: By Theorem 25[1] we have $\neg(A\vee B)\dashv\vdash\neg A\wedge\neg B$, and it is easy to prove $\neg\neg(A\vee B)\dashv\vdash\neg(\neg A\wedge\neg B)$ from Theorems 18[ ] and 25[2]. So $\neg(A\vee B)\models\!\dashv\neg A\wedge\neg B$ holds by Theorem 17[2].

Proof of [4]: Firtly, with Theorem 18[1], we have $\neg(A\wedge B)\models\!\dashv\neg(\neg\neg A\wedge\neg\neg B)$; secondly, we get, form Theorem 25[3], $\neg(\neg\neg A\wedge\neg\neg B)\models\!\dashv\neg\neg(\neg A\vee\neg B)$; then, form Theorem 18[1], $\neg\neg(\neg A\vee\neg B)\models\!\dashv\neg A\vee\neg B$, hence $\neg(A\ \ B)\models\!\dashv\neg A\vee\neg B$.

Theorem 26 MP:

[1] $A\vee B\models\!\dashv B\vee A$

[2] $A\wedge B\models\!\dashv B\wedge A$

[3] $A\vee(B\vee C)\models\!\dashv(A\vee B)\vee C$

[4] $A\wedge(B\wedge C)\models\!\dashv(A\wedge B)\wedge C$

Proof of [1]: (i) "$A\wedge B\dashv\vdash B\wedge A$" follows from Theorems 23[1] and [2].

(ii) First we prove $A\vee B\dashv\vdash B\vee A$

$\vdash$ (1) □□□□□ A

(2) □□□□□ $B\vee A$ (1)(MP24[2])

(3) □□□ B

(4) □□□ $B\vee A$ (3)(MP24[2])

(5) $A\vee B$

(6) $B\vee A$ (2)(4)(MP24[1])

Proof of $B\vee A\vdash A\vee B$ is just the same.

(iii) Now We prove $\neg(A\vee B)\dashv\vdash\neg(B\vee A)$

$\vdash$ (1) $\neg(A\vee B)$

(2) $\neg A\wedge\neg B$ (1)(MP25[1])

(3) $\neg B\wedge\neg A$ (2)above(i)

(4) $\neg(B\vee A)$ (3)(MP25[1])

Similarly we can get $\neg(B\vee A)\vdash\neg(A\vee B)$.

So by (ii)、(iii) above and Theorem 17[2] we have $A\vee B\models\!\dashv B\vee A$.

Proof of [2]: In the proof of Theorem 26[1] we have mentioned that $A\wedge B\dashv\vdash B\wedge A$ is obviously true and we have already proved $A\vee B\dashv\vdash B\vee A$. Thus $\neg(A\wedge B)\dashv\vdash\neg(B\wedge A)$ follows from Theorem 25[2], and by Theorem 17[2] we have $A\wedge B\models\!\dashv B\wedge A$.

Proof of [3]: (i) It is easy to prove $A\wedge(B\wedge C)\dashv\vdash(A\wedge B)\wedge C$ by Theorems 23[1] and [2].

(ii) Now we prove A ∨ (B ∨ C) ⊢⊣ (A ∨ B) ∨ C

⊢ (1) □□□□□□ B

(2) □□□□□ A ∨ B　　(1)(MP 24[2])

(3) □□□□□ (A ∨ B) ∨ C　　(2)(MP 24[2])

(4) □□□□ C

(5) □□□□ (A ∨ B) ∨ C　　(4)(MP 24[2])

(6) □□□ B ∨ C

(7) □□□ (A ∨ B) ∨ C　　(3)(5)(MP 24[1])

(8) □□ A

(9) □□ A ∨ B　　(8)(MP 24[2])

(10) □□ (A ∨ B) ∨ C　　(9)(MP 24[2])

(11) A ∨ (B ∨ C)

(12) (A ∨ B) ∨ C　　(7)(10)(MP 24[1])

Similary we can prove (A ∨ B) ∨ C ⊢ A ∨ (B ∨ C)

(iii) Then we prove ╕[A ∨ (B ∨ C)] ⊢⊣ ╕[(A ∨ B) ∨ C]

⊢ (1) ╕[A ∨ (B ∨ C)]

(2) ╕A ∧ ╕(B ∨ C)　　(1)(MP 25[1])

(3) ╕A ∧ (╕B ∧ ╕C)　　(2)(MP 25[3])

(4) (╕A ∧ ╕B) ∧ ╕C　　(3)above (i)

(5) ╕(A ∨ B) ∧ ╕C　　(4)(MP 25[3])

(6) ╕[(A ∨ B) ∨ C]　　(5)(MP 25[1])

Similarly we may prove ╕[(A ∨ B) ∨ C] ⊢ ╕[(A ∨ (B ∨ C)].

So by the above (ii)、(iii) and Theorem 17[2] we have A ∨ (B ∨ C) ⊨⫤ (A ∨ B) ∨ C

Proof of [4]: In proving Theorem 26[3] we have A ∧ (B ∧ C) ⊢⊣ (A ∧ B) ∧ C and A ∨ (B ∨ C) ⊢⊣ (A ∨ B) ∨ C,

then from these and by Theorems 25[2]、[4] we have ╕[A ∧ (B ∧ C)] ⊢⊣ ╕[(A ∧ B ∧ C].So A ∧ (B ∧ C) ⊨⫤ (A ∧ B) ∧ C, by Theorem 17[2].

Theorem 27 MP:

[1] A ⊢⊣ ¬~A ∧ ¬╕A　　[4] ¬A ⊨⫤ ~A ∨ ╕A

[2] ╕A ⊢⊣ ¬A ∧ ¬~A　　[5] ¬╕A ⊨⫤ A ∨ ~A

[3] ~A ⊨⫤ ¬A ∧ ¬╕A　　[6] ¬~A ⊨⫤ A ∨ ╕A

Proof of [1]: Using Theorems 23[1]、[2]and (Y).

Proof of [2]: Using Theorems 23[1]、[2]and ($Y_{╕}$).

Proof of [3]: (i) It is easy to prove ~A ⊢⊣ ¬A ∧ ¬╕A by Theorems 23[1]、[2]and ($Y_{\sim}$).

(ii) Now we prove ╕(¬A ∧ ¬╕A) ⊢⊣ ╕~A

⊢ (1) ╕(¬A ∧ ¬╕A)

(2) ╕¬A ∨ ╕¬╕A　　(1)(MP 25[2])

(3) $\exists\neg\exists A \vee \exists\neg A$ (2)(MP 26[1])

(4) $\exists\exists\neg\exists A \to \exists\neg A$ (3)(D($\vee$))

(5) $\neg\exists A \to \exists\neg A$ (4)(MP 18[1])

(6) $\neg\exists\neg A$ (MP 15[2])

(7) $\exists\neg\exists A$ (5)(6)(MP 8[1])

(8) $\neg\exists\neg\exists A$ (MP 15[2])

(9) $\exists\sim A$ (7)(8)(MP 2[1])

and $\exists\sim A \vdash \exists(\neg A \wedge \neg\exists A)$ follows immediately from Theorem 15[3].

So by (i)、(ii)and Theorem 17[2] we have $\sim A \models\!\dashv \neg A \wedge \neg\exists A$.

Proof of [4]: (i) First we have $\neg A \vdash\!\dashv \sim A \vee \exists A$

$\vdash$ (1) $\neg A$

(2) □□□□□ A

(3) □□□□□ $\sim A$ (1)(2)(MP 2[1])

(4) □□□ $\sim A$

(5) □□□ $\sim A$ (4)($\in$)

(6) $A \to \sim A$ (3)(5)($\to_+$)

(7) $\exists\exists A \to \sim A$ (6)(MP 18[1])

(8) $\exists A \vee \sim A$ (7)(D($\vee$))

(9) $\sim A \vee \exists A$ (8)(MP 26[1])

and $\sim A \vee \exists A \vdash \neg A$ holds by ($Y_\sim$)、($Y_\exists$) and Theorem 24[1].

(ii) "$\exists\neg A \vdash \exists(\sim A \vee \exists A)$" is obviously true by Theorem 15[4],and it is easy to prove $\exists(\sim A \vee \exists A) \vdash \exists\neg A$ by Theorems 25[3]、23[1]and 15[3], hence $\exists\neg A \vdash\!\dashv \exists(\sim A \vee \exists A)$.

So from (i)、(ii)and Theorem 17[2] we have $\neg A \models\!\dashv \sim A \vee \exists A$.

Proof of [5]: First by Theorem 27[4] we have $\neg\exists A \models\!\dashv \sim\exists A \vee \exists\exists A$, and byTheorem 18[1], $\sim\exists A \vee \exists\exists A \models\!\dashv \sim\exists A \vee A$, then by Theorem 18[3] $\sim\exists A \vee A \models\!\dashv \sim A \vee A$, by Thenrem 26[1] $\sim A \vee A \models\!\dashv A \vee \sim A$, hence $\neg\exists A \models\!\dashv A \vee \sim A$.

Proof of [6]: "$\neg\sim A \vdash A \vee \exists A$" is easy to be verified by Theorem 2[1]、D($\vee$) and Theoıem 26[2], and so is $A \vee \exists A \vdash \neg\sim A$ by Theorem 24[1]、(Y)and ($Y_\exists$), hence $\neg\sim A \vdash\!\dashv A \vee \exists A$. On the other hand, by Theorems 15[2]and 2[1] we have $\exists\neg\sim A \vdash \exists(A \vee \exists A)$. Now let us prove

$$\exists(A \vee \exists A) \vdash \exists\neg\sim A,$$

(1) $\exists(A \vee \exists A)$

(2) $\exists A \wedge \exists\exists A$ (1)(MP 25[1])

(3) $\exists A$ (2)(MP 23[1])

(4) $\exists\exists A$ (2)(MP 23[1])

(5) A (4)(MP 3[1])

(6) $\neg A$ (3)($Y_\exists$)

(7) $\exists\neg\sim A$ (5)(6)(MP 2[1])

thus we have also proved $\exists\neg\sim A \vdash\!\dashv \exists(A \vee \exists A)$. Therefore, we have $\neg\sim A \models\!\dashv A \vee \exists A$ by Theorem 17[2].

Theorem 28 $\sim\sim A \vDash\!\dashv \neg\sim A \vDash\!\dashv A\rightarrow A$

Proof By Theorem 18[2] $\sim\sim A \vDash\!\dashv A\rightarrow A$, and by Theorem 27[6] $\neg\sim A \vDash\!\dashv A\vee\exists A$, then by Theorem 26[1] $A\vee\exists A \vDash\!\dashv \exists A\vee A$, morever, by D($\vee$)and Theorem 18[1] $\exists A\vee A \vDash\!\dashv A\rightarrow A$, hence $\sim\sim A \vDash\!\dashv \neg\sim A \vDash\!\dashv A\rightarrow A$.

Theorem 29 MP:

[1] $A\vee(B\wedge C) \vDash\!\dashv (A\vee B)\wedge(A\vee C)$

[2] $A\wedge(B\vee C) \vDash\!\dashv (A\wedge B)\vee(A\wedge C)$

Proof of [1]、[2]: (i) Prove $A\vee(B\wedge C) \vdash\!\dashv (A\vee B)\wedge(A\vee C)$

$\vdash$ (1) □□□□ A

(2) □□□□ $A\vee B$ (1)(MP 24[2])

(3) □□□□ $A\vee C$ (1)(MP 24[2])

(4) □□□□ $(A\vee B)\wedge(A\vee C)$ (2)(3)(MP 23[2])

(5) □□ $B\wedge C$

(6) □□ B (5)(MP 23[1])

(7) □□ C (5)(MP 23[1])

(8) □□ $A\vee B$ (6)(MP 24[2])

(9) □□ $A\vee C$ (7)(MP 24[2])

(10) □□ $(A\vee B)$ $(A\vee C)$ (8)(9)(MP 23[2])

(11) $A\vee(B\wedge C)$

(12) $(A\vee B)\wedge(A\vee C)$ (4)(10)(MP 24[1])

$\dashv$ (1) $(A\vee B)\wedge(A\vee C)$

(2) □□□□□ $\exists A$

(3) □□□□□ $A\vee B$ (1)(MP 23[1])

(4) □□□□□ $\exists A\rightarrow B$ (3)(D($\vee$))

(5) □□□□□ B (2)(4)($\rightarrow_-$)

(6) □□□□□ C similar to (5)

(7) □□□□□ $B\wedge C$ (5)(6)(MP 23[2])

(8) □□ $\sim\exists A$

(9) □□ $B\wedge C$ similar to (7)

(10) $\exists A\rightarrow(B\wedge C)$ (7)(9)($\rightarrow_+$)

(11) $A\vee(B\wedge C)$ (10)(D($\vee$))

(ii) Now we prove $A\wedge(B\vee C) \vdash\!\dashv (A\wedge B)\vee(A\wedge C)$.

"$A\wedge(B\vee C) \vdash (A\wedge B)\vee(A\wedge C)$" follows from Theorem 11[5], and then $(A\wedge B)\vee(A\wedge C) \vdash A\wedge(B\vee C)$ holds by Theorems24[1]、[2]and 23[1]、[2].

(iii) Prove $\exists[A\vee(B\wedge C)] \vdash\!\dashv \exists[(A\vee B)\wedge(A\vee C)]$.

Using Theorem 25[1]、[3]、[4]and the above (ii).

(iv) At last we prove $\exists[A\wedge(B\vee C)] \vdash\!\dashv \exists[(A\wedge B)\vee(A\wedge C)]$.

Using Theorem 25[1]、[2]、[3]、[4]and above (i).

Therefore from (i), (iii) and Theorem 17[2] we have $A\vee(B\wedge C)\models\!\dashv(A\vee B)\wedge(A\vee C)$, and from (ii), (iv) and Theorem 17[2] we have $A\wedge(B\vee C)\models\!\dashv(A\wedge B)\vee(A\wedge C)$.

Theorem 30 MP:

[1] $A\models\!\dashv A\vee A$ [2] $A\models\!\dashv A\wedge A$

Proofs of [1], [2]: It is easy to prove the following (i), (ii), (iii), (iv) in turn

(i) $A\vdash\!\dashv A\vee A$ (ii) $A\vdash\!\dashv A\wedge A$ (iii) $\urcorner A\vdash\!\dashv\urcorner(A\vee A)$ (iv) $\urcorner A\vdash\!\dashv\urcorner(A\wedge A)$

then by combining (i) and (iii), (ii) and (iv), respectively, and using Theorem 17[2], we have Theorem 30.

Theorem 31 MP:

[1] $A\rightarrow(B\wedge C)\models\!\dashv(A\rightarrow B)\wedge(A\rightarrow C)$ [2] $A\rightarrow(B\vee C)\models\!\dashv(A\rightarrow B)\vee(A\rightarrow C)$

[3] $(A\wedge B)\rightarrow C\models\!\dashv(A\rightarrow C)\vee(B\rightarrow C)$ [4] $(A\vee B)\rightarrow C\models\!\dashv(A\rightarrow C)\wedge(B\rightarrow C)$

Proof of [1]: From Theorem 18[1] $A\rightarrow B\wedge C\models\!\dashv\urcorner\urcorner A\rightarrow B\wedge C$, and from D($\vee$) $\urcorner\urcorner A\rightarrow B\wedge C\models\!\dashv\urcorner A\vee(B\wedge)$, and from Theorem 29[1] $\urcorner A\vee(B\wedge C)\models\!\dashv(\urcorner A\vee B)\wedge(\urcorner A\vee C)$, then from D($\vee$)

$(\urcorner A\vee B)\wedge(\urcorner A\vee C)\models\!\dashv(\urcorner\urcorner A\rightarrow B)\wedge(\urcorner\urcorner A\rightarrow C)$, also from Theorem 18[1] $(\urcorner\urcorner A\rightarrow B)\wedge(\urcorner\urcorner A\rightarrow C)\models\!\dashv(A\rightarrow B)\wedge(A\rightarrow C)$, thus we have $A\rightarrow B\wedge C\models\!\dashv(A\rightarrow B)\wedge(A\rightarrow C)$.

Proof of [2]: Following the argument of [1] and in turn using Theorem 18[1], D($\vee$), Theorem 30[1], Theorems 26[2], [4], D($\vee$) and Theorem 18[1].

Proof of [3]: Following the argument of [1] and in turn using Theorem 18[1] D($\vee$), Theorem 25[4], Theorem 30[1], theorem 26[2], [4], D[$\vee$], and Theorem 18[1].

Proof of [4]: Following the argument of [1] and in turn using Theorem 18[1] D($\vee$), Theorem 25[3], Theorem 26[2], Theorem 29[1], Theorem 26[2], D($\vee$) and Theorem 18[1]..

Theorem 32 MP:

[1] $A\vee(B\wedge\urcorner B)\vdash\!\dashv A$ [2] $A\wedge(B\vee\urcorner B)\vdash A$

Proof of [1]: Using Theorem 11[5] we have $A\vee(B\wedge\urcorner B)\vdash A$, and $A\vdash A\vee(B\wedge\urcorner B)A$ is immediately got by Theorem 24[2].

Proof of [2]: It is immedialely got by Theorem 23[1].

By the way, we can not have $A\vdash A\wedge(B\vee\urcorner B)$.

Theorem 33: If Γ, $A\vdash C$ and Γ, $B\vdash C$, then Γ, $A\vee B\vdash C$.

Proof If $\Gamma=\varnothing$, then the theorem is just Theorem 24[1] which has been proved before. Thus let $\Gamma=A_1, A_2, A_3, \cdots, A_n$, and write briefly $A_1\wedge A_2\wedge\cdots\wedge A_n$ as $\Gamma_\wedge$, we have:

(1) $\Gamma_\wedge \wedge A \vdash \Gamma, A$ (MP 23[1])

(2) $\Gamma_\wedge \wedge B \vdash \Gamma, B$ (MP 23[1])

(3) $\Gamma, A \vdash C$ hypothesis

(4) $\Gamma, B \vdash C$ hypothesis

(5) $\Gamma_\wedge \wedge A \vdash C$ (1)(3)(τ)

(6) $\Gamma_\wedge \wedge B \vdash C$ (2)(4)(τ)

(7) $(\Gamma_\wedge \wedge A) \vee (\Gamma_\wedge \wedge B) \vdash C$ (5)(6)(MP 24[1])

(8) $\Gamma, A \vee B \vdash \Gamma_\wedge \wedge (A \vee B)$ (MP 23[2])

(9) $\Gamma_\wedge \wedge (A \vee B) \vdash (\Gamma_\wedge \wedge A) \vee (\Gamma_\wedge \wedge B)$ (MP 29[2])

(10) $\Gamma, A \vee B \vdash C$ (8)(9)(7)(τ)

Theorem 34 MP:

[1] $A, B \vdash A \leftrightarrow B$

[2] $A \leftrightarrow B \dashv\vdash B \leftrightarrow A$

[3] $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$

[4] $A \leftrightarrow B \dashv\vdash \neg A \leftrightarrow \neg B$

[5] $\neg A, \neg B \vdash A \leftrightarrow B$

[6] $A \leftrightarrow B \dashv\vdash (\neg A \vee B) \wedge (\neg B \vee A)$

[7] $A \leftrightarrow B \dashv\vdash (A \wedge B) \vee (\neg A \neg B)$

[8) $A \leftrightarrow B \vdash \neg \sim A, \neg \sim B$

Proof of [1]: By Theorem 1[2]、Theorem 23[2] and D($\wedge$).

Proof of [2]: From D($\wedge$) $A \leftrightarrow B \dashv\vdash (A \rightarrow B) \wedge (B \rightarrow A)$, and by Theorem 26[2] $(A \rightarrow B) \wedge (B \rightarrow A) \dashv\vdash (B \rightarrow A) (A \rightarrow B)$, then by D($\wedge$) $(B \rightarrow A) \wedge (A \rightarrow B) \dashv\vdash B \leftrightarrow A$, hence $A \leftrightarrow B \dashv\vdash B \leftrightarrow A$.

Proof of [3]: Using D($\leftrightarrow$)、Theorems 23[1]、[2] and Theorem 1[3].

Proof of [4]: Following the argument of [2] and in turn using D($\leftrightarrow$) Theorem 22、D($\leftrightarrow$)、Theorem 34[2].

Proof of [5]: By Theorem 34[1]、[2].

Proof of [6]: Following the argumunt of [2] and in turn using D($\leftrightarrow$) Theorem 18[1]、D($\vee$).

Proof of [7]: (i) First we prove $A \leftrightarrow B \vdash (A \wedge B) \vee (\neg A \wedge \neg B)$

$\vdash$ (1) $A \leftrightarrow B$

(2) $(\neg A \vee B) \wedge (\neg B \vee A)$ (1)(MP 34[6])

(3) $(\neg A \wedge \neg B) \vee (B \wedge A) \vee (\neg A \wedge A) \vee (B \wedge \neg B)$ (2)using MP29[2]twice

(4) $(A \wedge B) \vee (\neg A \wedge \neg B)$ (3)using MP32[1]tiwice

By going back we can prove $(A \wedge B) \vee (\neg A \wedge \neg B \vdash A \leftrightarrow B$.

(ii) Now we prove $\neg(A \leftrightarrow B) \vdash \neg[(A \wedge B) \vee (\neg A \wedge \neg B)]$

$\vdash$ (1) $\neg(A \leftrightarrow B)$

(2) $\neg[(\neg A \vee B) \wedge (\neg B \vee A)]$ (1)(MP 34[6])

(3) $\neg(\neg A \vee B) \vee \neg(\neg B \vee A)$ (2)(MP 25[2])

(4) $(\neg\neg A \wedge \neg B) \vee (\neg\neg B \wedge \neg A)$ (3)(MP 25[1])

(5) $(A \wedge \neg B) \vee (B \wedge \neg A)$ (4)(MP 18[1])

(6) $(A \vee B) (B \vee \neg B) \wedge (A \vee \neg A) \wedge (\neg B \wedge \neg A$ (5)using MP 29[1] twice

(7) $(A \vee B) \wedge (\neg A \vee \neg B)$ (6) using MP 32[2] twice

(8) $(\neg\neg A \vee \neg\neg B) \wedge (\neg A \vee \neg B)$ (7)(MP 18[1])

(9) $\neg(\neg A \wedge \neg B) \wedge \neg(A \wedge B)$ (8)(MP 25[2])

(10) $\neg[(\neg A \wedge \neg B) \vee (A \wedge B)]$ (9)(MP 25[1])

(11) $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)]$ (10)(MP 26[1])

then going back we prove $\neg[(A \wedge B) \vee (\neg A \wedge \neg B)] \vdash \neg(A \leftrightarrow B)$.

Hence from (i), (ii) and Theorem 17[2] we have $A \leftrightarrow B \vDash\!\!\dashv (A \wedge B) \vee (\neg A \wedge \neg B)$.

Proof of [8]: By Theorems 34[7] and 24[1].

Theorem 35 MP:

[1] $\neg(A \leftrightarrow B) \vDash\!\!\dashv (A \wedge \neg B) \vee (\neg A \wedge B)$

[2] $\neg(A \leftrightarrow B) \vDash\!\!\dashv A \leftrightarrow \neg B \vDash\!\!\dashv \neg A \leftrightarrow B$

[3] $A,\ \neg B \vdash \neg(A \leftrightarrow B)$

[4] $\neg A,\ B \vdash \neg(A \leftrightarrow B)$

Proof of [1]: From Theorem 34[6]

$$\neg(A \leftrightarrow B) \vDash\!\!\dashv \neg[(\neg A \vee B) \wedge (\neg B \vee A)],$$

from Theorem 25[4]

$$\neg[(\neg A \vee B) \wedge (\neg B \vee A)] \vDash\!\!\dashv \neg(\neg A \vee B) \vee \neg(\neg B \vee A),$$

from Theorem 25[3]

$$\neg(\neg A \vee B) \vee \neg(\neg B \vee A) \vDash\!\!\dashv (\neg\neg A \wedge \neg B) \vee (\neg\neg B \wedge A),$$

from Theorem 18[1]

$$(\neg\neg A \wedge \neg B) \vee (\neg\neg B \wedge \neg A) \vDash\!\!\dashv (A \wedge \neg B) \vee (B \wedge \neg A),$$

from Theorem 26[1]

$$(A \wedge \neg B) \vee (B \wedge \neg A) \vDash\!\!\dashv (A \wedge \neg B) \vee (\neg A \wedge B),$$

hence $\neg(A \leftrightarrow B) \vDash\!\!\dashv (A \wedge \neg B) \vee (\neg A \wedge B)$.

Proof of [2]: Following the argument of [1] and in turn using Theorem 35[1], Theorem 18[1], Theorem 34[7] we can prove $\neg(A \leftrightarrow B) \vDash\!\!\dashv A \leftrightarrow \neg B$. In the just same way we get that $\neg(B \leftrightarrow A) \vDash\!\!\dashv B \leftrightarrow \neg A$, and from Theorem 34[2] we get $\neg(A \leftrightarrow B) \vDash\!\!\dashv \neg A \leftrightarrow B$. By combining with the above the desired comclusion is at hand.

Proof of [3]: By Theorems 34[1] and 35[2].

Proof of [4]: By Theorems 34[1] and 35[2], too.

Theorem 36 MP:

[1] $\sim A \vdash \sim(A \leftrightarrow B)$ [2] $\sim B \vdash \sim(A \leftrightarrow B)$ [3] $\sim(A \leftrightarrow B) \vDash\!\!\dashv \sim A \vee \sim B$

Proof of [1]: By Theorem 11[5].

Proof of [2]: By Theorem 36[1] and 34[2].

Proof of [3]: (i) First we prove $\sim(A \leftrightarrow B) \vdash\!\dashv \sim A \vee \sim B$

$\vdash$ (1) $\sim(A \leftrightarrow B)$

(2) □□□□□□□□□□ $\neg\sim A$

(3) □□□□□□□□□□ ㄱ $\neg\sim A$ (MP 15[1])

(4) □□□□□□□□□□ $\sim B$ (2)(3)(MP 2[1])

(5) □□□□□□ A

(6) □□□□□□□□ B

(7) □□□□□□□□ A↔B (5)(6)(MP34[1])

(8) □□□□□□□□┐(A↔B) (1)(Y~)

(9) □□□□□□□ ㅋB

(10) □□□□□□□ ㅋ(A↔B) (5)(9)(MP 35[3])

(11) □□□□□□□┐ ㅋ(A↔B) (1)(Y~)

(12) □□□□□□ ~B (7)(8)(10)(11)(MP 6[2])

(13) □□□ ㅋA

(14) □□□□□ ㅋB

(15) □□□□□ A↔B (13)(14)(MP 34[5])

(16) □□□□□┐(A↔B) (1)(Y~)

(17) □□□□ B

(18) □□□□ ㅋ(A↔B) (13)(17)(MP 35[4])

(19) □□□□┐ ㅋ(A↔B) (1)(Y~)

(20) □□□ ~B (15)(16)(18)(19)(MP 6[2])

(21) □ ~ㅋ~A

(22) □ ~~A (21)(MP 18[3])

(23) □┐ ~A (22)(Y~)

(24) □ A∨ㅋA (23)(MP 27[6])

(25) □ ~B (12)(20)(MP 24[1])

(26) ㅋ~A→~B (4)(25)(→+)

(27) ~A∨~B (26)(D(∨))

⊣(1) □□□□□ ~A

(2) □□□□□ ~(A↔B) (1)(MP 36[1])

(3) □□□ ~B

(4) □□□ ~(A↔B) (3)(MP 36[2])

(5) ~A∨~B

(6) ~(A↔B) (2)(4)(MP 24[1])

(ii) Now we prove ㅋ~(A↔B) ⊢⊣ ㅋ(~A∨~B)

⊢ (1) ㅋ~(A↔B) (2) ┐ ㅋ~(A↔B) (MP 15[1]) (3) ㅋ(~A∨~B) (1)(2)(MP 2[1])

⊣ (1) ㅋ(~A∨~B)

(2) ㅋ~A∧ㅋ~B (1)(MP 25[1]) (4) ┐ ㅋ~A (MP 15[1])

(3) ㅋ~A (2)(MP 23[1]) (5) ㅋ(~A∨~B) (3)(4)(MP2[1])

Hence from (i), (ii) and Theorem 17[2] we have ~(A↔B) ⊨⊣ ~A∨~B.

Theorm 37 MP:

[1] ~(A∨B) ⊢ ~A∨~B [2] ~(A∧B) ⊢ ~A∨~B

Proof of[1]:

(1) ~(A ∨ B)
(2) ~(╕A → B)   (1)(D(∨))
(3) □□□□□□□□□ ╕A
(4) □□□□□□□□□□ ~B   (2)(3)(MP 12[4])
(5) □□□□□□□ A
(6) □□□□□□□ ╕╕A   (5)($╕╕_{+}$)
(7) □□□□□□□ ╕A → B   (6)(MP 5[1])
(8) □□□□□□□ㄱ(╕A → B)   (2)($Y_{\sim}$)
(9) □□□□□□□ ~B   (7)(8)(MP 2[1])
(10) □□□□□ ~╕~A
(11) □□□□□ ~~A   (10)(MP 18[3])
(12) □□□□□ A → A   (11)(MP 18[2])
(13) □□□□□ ╕╕A → A   (12)(MP 18[1])
(14) □□□□□ ╕A ∨ A   (13)(D(∨))
(15) □□□□□ ~B   (4)(9)(MP 33)
(16) □□□ ╕~A
(17) □□□ㄱ ╕~A   (MP 15[1])
(18) □□□ ~B   (16)(17)(MP 2[1])
(19) ╕~A → ~B   (15)(18)($\rightarrow_{+}$)
(20) ~A ∨ ~B   (19)(D(∨))

Proof of [2]:

(1) ~(A ∧ B)
(2) ~╕(A → ╕B)   (1)(D(∧))
(3) ~(A → ╕B)   (2)(MP 18[3])
(4) ~(╕╕A → ╕B)   (3)(MP [1])
(5) ~(╕A ∨ ╕B)   (4)(D(∨))
(6) ~╕A ∨ ~╕B   (5)(MP37[1])
(7) ~A ∨ ~B   (6)(MP18[3])

Theorem 38 MP:

[1] ~(A ∨ B) ⊨⫤ (~A ∧ ~B) ∨ (~A ∧ ╕B) ∨ (╕A ∧ ~B)

[2] ~(A ∧ B) ⊨⫤ (~A ∧ ~B) ∨ (~A ∧ B) ∨ (A ∧ ~B)

Proof of [1]: (i) From Theorem 24[1] we have

~(A ∨ B) ⊢⊣ (~A ∧ ~B) ∨ (~A ∧ ╕B) ∨ (╕A ∧ ~B).

(ii) From Theorems 15[1] and 2[1] we have immediately that

╕~(A ∨ B) ⊢ ╕[(~A ∧ ~B) ∨ (~A ∧ ╕B) ∨ (╕A ∧ ~B)],

and from Theorem 24[1] we see that

╕[(~A ∧ ~B) ∨ (~A ∧ ╕B) ∨ (╕A ∧ ~B)] ⊢ ╕~(A ∨ B).

Hence we conclude, from (i)、(ii) and Theorem 17[2], that

~(A ∨ B) ⊨⫤ (~A ∧ ~B) ∨ (~A ∧ ╕B) ∨ (╕A ∧ ~B).

Proof of [2]: From Theorem 18[1] we have

$$\sim(A\wedge B)\vDash\!\dashv\sim(\neg\neg A\wedge\neg\neg B),$$

by Theorem 25[3] we have

$$\sim(\neg\neg A\wedge\neg\neg B)\vDash\!\dashv\sim\neg(\neg A\vee\neg B).$$

by Theorem 18[3] we have

$$\sim\neg(\neg A\vee\neg B)\vDash\!\dashv\sim(\neg A\vee\neg B),$$

but by Theorem 38[1] we have

$$\sim(\neg A\vee\neg B)\vDash\!\dashv(\sim\neg A\wedge\sim\neg B)\vee(\sim\neg A\wedge\neg\neg B)\vee(\neg\neg A\wedge\sim\neg B),$$

and by Theorems 18[1]、[3] again we have

$$(\sim\neg A\wedge\sim\neg B)\vee(\sim\neg A\wedge\neg\neg B)\vee(\neg\neg A\wedge\sim\neg B)\vDash\!\dashv$$
$$\vDash\!\dashv(\sim A\wedge\sim B)\vee(\sim A\wedge B)\vee(A\wedge\sim B),$$

hence we have

$$\sim(A\wedge B)\vDash\!\dashv(\sim A\wedge\sim B)\vee(\sim A\wedge B)\vee(A\wedge\sim B).$$

Theorem 39 MP:

[1] $\sim A\vee\sim B\vDash\!\dashv\sim(A\vee B)\vee\sim(A\wedge B)$ [2] $\sim A\wedge\sim B\vDash\!\dashv\sim(A\vee B)\wedge\sim(A\wedge B)$

Proof of [1]: (i) First we prove $\sim A\vee\sim B\vdash\!\dashv\sim(A\vee B)\vee\sim(A\wedge B)$

$\vdash$ (1) $\Box\Box\Box\Box\sim A$

(2) $\Box\Box\Box\Box\Box\Box\Box B$

(3) $\Box\Box\Box\Box\Box\Box\Box\sim(A\ \ B)$ (1)(2)(MP 23[5])

(4) $\Box\Box\Box\ \ \Box\Box\Box\sim(A\vee B)\vee\sim(A\wedge B)$ (3)(MP 24[2])

(5) $\Box\Box\Box\Box\Box\Box\sim B$

(6) $\Box\Box\Box\Box\Box\Box\sim(A\wedge B)$ (1)(5)(MP 23[6])

(7) $\Box\Box\Box\Box\Box\Box\sim(A\vee B)\vee\sim(A\wedge B)$ (6)(MP 24[2])

(8) $\Box\Box\Box\Box\Box\ \neg B$

(9) $\Box\Box\Box\Box\Box\sim(A\vee B)$ (1)(8)(MP 24[5])

(10) $\Box\Box\Box\Box\Box\sim(A\vee B)\vee\sim(A\wedge B)$ (9)(MP 24[2])

(11) $\Box\Box\Box\Box\sim(A\vee B)\vee\sim(A\wedge B)$ (4)(7)(10)(MP 11[5])

(12) $\Box\Box\sim B$

(13) $\Box\Box\sim(B\vee A)\vee\sim(B\wedge A)$ similar to (11)

(14) $\Box\Box\sim(A\vee B)\vee\sim(A\wedge B)$ (13)(MP 26[1]、[2])

(15) $\sim A\vee\sim B$

(16) $\sim(A\vee B)\vee\sim(A\wedge B)$ (11)(14)(MP 24[1])

then using Theorems 37[1]、[2] and Theorem 24[1] $\sim(A\vee B)\vee\sim(A\wedge B)\vdash\sim A\vee\sim B$,

(ii) By Theorems 25[1]、23[1] and 15[3]

we can prove $\neg(\sim A\vee\sim B)\vDash\!\dashv\neg[\sim(A\vee B)\vee\sim(A\wedge B)]$. Hence by (i)、(ii) and Theorem 17[2] we have $\sim A\vee\sim B\vDash\!\dashv\sim(A\vee B)\vee\sim(A\wedge B)$.

Proof of [2]: (i) First we prove $\sim A\wedge\sim B\vdash\!\dashv\sim(A\vee B)\wedge\sim(A\wedge B)$

(1) $\sim A\wedge\sim B$

(2) $\sim A$ (1)(MP 23[1])

(3) $\sim B$ (1)(MP 23[1])

(4) $\sim(A\vee B)$ (2)(3)(MP 24[6])

(5) ~(A B)   (2)(3)(MP 23[6])

(6) ~(A ∨ B) ~(A B)   (4)(5)(MP 23[2])

—| (1) ~(A ∨ B) ~(A B)

(2) □□□□□□□□ A

(3) □□□□□□□□ A ∨ B   (2)(MP 21[2])

(4) □□□□□□□□ ~(A ∨ B)   (1)(MP 23[1])

(5) □□□□□□□□ ¬(A ∨ B)   (4)(Y~)

(6) □□□□ ╕A

(7) □□□□ ╕(A ∧ B)   (6)(MP 23[3])

(8) □□□□ ~(A ∧ B)   (1)(MP 23[1])

(9) □□□□ ¬ ╕(A ∧ B)   (8)(Y~)

(10) ~A   (3)(5)(7)(9)(MP 6[2])

(11) ~B   similar to (10)

(12) ~A ∧ ~B   (10)(11)(MP 23(2])

(ii) Now we prove ╕(~A ∧ ~B) ├┤ ╕[~(A ∨ B) ∧ ~(A ∧ B)]

├ (1) ╕(~A ∧ ~B)

(2) ╕~A ∨ ╕~B   (1)(MP 25[2])

(3) □□□□□ ╕~A

(4) □□□□□ ╕[~(A ∨ B) ∧ ~(A ∧ B)]   (3)(MP 15[3])

(5) □□□ ╕~B

(6) □□□ ╕[~(A ∨ B) ∧ ~(A ∧ B)]   (5)(MP 15[3])

(7) ╕[~(A ∨ B) ∧ ~(A ∧ B)]   (4)(6)(MP 24[1])

and by Theorems 25[2], 15[3] and 24[1], ╕[~(A ∨ B) ∧ ~(A ∧ B)] ├ ╕(~A ∧ ~B).
hence from (i)、(ii) and Theorem 17[2] we have ~A ∧ ~B ╞╡ ~(A ∨ B) ∧ ~(A ∧ B);

Theorem 40 MP:

[1] $A_1 \to B_1$, $A_2 \to B_2 \vdash A_1 \wedge A_2 \to B_1 \wedge B_2$

[2] $A_1 \to B_1$, $A_2 \to B_2 \vdash A_1 \vee A_2 \to B_1 \vee B_2$

Proof of [1]:Using Theorems 23[2]、[1]、Theorem 38[1], Theorem 21[1]and $(\to_+)$.

Proof of [2] Using Theorem 24[1]、Theorem 37[1] and $(\to_+)$.

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