Journal of Mathematical Research and Exposition
Vol. 10, No.2, May, 1990

The Spectrum of a Noetherian Ring*

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The concept of generalized primary rings was introduced by Satyanarayana [1]. In a similar fashion, we define a generalized semi-primary ring to be a ring of which each ideal is semi-primary. It follows from [1] that there exist at most tmo prime ideals in any generalized primary (or generalized semi-primary) ring R with $\dim R \leqslant 1$. In this paper we shall consider the case $\dim R \geqslant 2$ and discuss the relations between its prime ideals of height i and those either of height $i-1$ or of height $i+1$.

We shall use R to designate a commutative ring with an identity and Spec (R), its spectrum, i.e., the set of all prime ideals of R.

Theorem 1 Let (R, m) be a noetherian local domain with $\dim R = 2$ and let $(P_i | i \in V)$ be the set of all prime ideals of R of height $h(P_i) = 1$. Then

(1) $\bigcup_{i \in V} P_i = m$ and V is infinite;

(2) $\bigcap_{i \in V} P_i = (0)$.

Proof That $\bigcup_{i \in V} P_i \subseteq m$ is obvions. If $\bigcup_{i \in V} P_i \neq m$, there exists $x \in m$, $x \bar{\in} \bigcup_{i \in V} P_i$ and $(x) \subseteq m$. Since m is the only prime ideal of R containing (x), we have $\sqrt{(x)} = m$. The ideal (x) is m-primary, because m is a maximal ideal of R. From the dimension theory of noetherian ring we know that the height $h(m) = 1$ This leads a contradiction. Thus $\bigcup_{i \in V} P_i = m$. If V is finite, then there exists $j \in V$ such that $m = P_j$ which contradicts the assumption $\dim R = 2$. hence (1) holds.

Next, suppose $\bigcap_{i \in V} P = I \neq (0)$. Since I is an ideal of the noetherian ring, there exists a primary factorization of I. Hence the number of prime ideals belonging to I is finite. Let $q_1, q_2, \cdots, q_n$ be the minimal prime ideals belonging to I, it is clear that $h(q_j) = 1$, $j = 1, 2, \cdots, n$. Since $P_i \supseteq I$ for each $i \in V$, there exists an integer j $(1 \leqslant j \leqslant n)$ such that $P_i = q_j$. Therefore V is finite. This contradiction shows the validity of (2).

Theorem 2 Let R be a noetherian ring of dimension n $(2 \leqslant n < \infty)$ and let

* Received Apr. 16, 1988

$P_0 \subset P_1 \subset \cdots \subset P_k$ be a maximal prime chain of R. Then to each integer i with $0 < i < k$, the set $\Sigma_i = \{P | P \in \mathrm{Spec}(R),\ P_{i-1} \subset P \subset P_{i+1}\}$ is infinite and $\bigcap_{P \in \Sigma_i} P = P_{i-1}$ $\bigcup_{P \in \Sigma_i} P_i = P_{i+1}$. In particular when n is finite, by taking $k = n$, then each noetherian ring R of dimension n contains an infinite number of prime ideals of height i $(0 < i < n)$.

Proof Since there exists a bijection between the set $\{P | P$ is a prime ideal of R/P_{i-1} and $h(P) = 1\}$ and the set $\{P | P$ is a prime ideal of R with height i such that $P \supset P_{i-1}\}$, P_{i+1}/P_{i-1} is a prime ideal of R/P_{i-1} of height 2. Localizing, $(R/P_{i-1})_{P_{i+1}/P_{i-1}}$ is a local domain of dimension 2. Consequently, there is a bijection between the set $\{P | P$ is a prime ideal of $(R/P_{i-1})_{P_{i+1}/P_{i-1}}$ and $h(P) = 1\}$ and the set $\{P | P$ is a prime ideal of R, $P_{i-1} \subset P \subset P_{i+1}$, $h(P) = i\}$. The rest of the proof follows immediately from Theorem 1.

Corollary 1 Let R be the same as in Theorem 2. Then

(1) Each prime ideal $m \in \mathrm{Spec}(R)$ with $1 < h(m) = k$ is the union of all prime ideals of height j $(0 < j < k)$ of R contained in m.

(2) $\sqrt{(0)}$ equals to the intersection of all prime ideals of R of height i $(0 < i < n)$. If $\bigcap_{P \in \max(R)} P = \sqrt{(0)}$, then R is not semi-local.

Proof (1) For each i $(1 < i < k)$ and $P \in \mathrm{Spec}(R)$ with $h(P) = i$, let $\{(P_a | a \in V\} = \{q \in \mathrm{Spec}(R) | q \subset P,\ h(q) = i - 1\}$. Then, by Theorem 2, we have $P = \bigcup_{a \in V} P_a$, hence

$$\bigcup_{\substack{P \in \mathrm{Spec}(R) \\ h(P) = i-1}} P = \bigcup_{\substack{P \in \mathrm{Spec}(R) \\ h(P) = i}} P.$$ This proves the first part of the corollary.

(2) Suppose that $\{P_i | i = 1, 2, \cdots, l\}$ is the set of all minimal prime ideals of R, then $\bigcap_{\substack{P \supset P_i \\ h(P) = 1}} P = P_i$ for each i. Hence $\bigcap_{\substack{P \in \mathrm{Spec}(R) \\ h(P) = 1}} P = \bigcap_{i=1}^{l} P = \sqrt{(0)}$. Similarly we have

$$\bigcap_{\substack{P \in \mathrm{Spec}(R) \\ h(P) = j}} P = \sqrt{(0)} \quad (0 < j < n).$$

Let $(R, m_1, m_2, \cdots, m_k)$ be a semi-local ring and $\bigcap_{i=1}^{k} m_i = \sqrt{(0)}$. Then there exists a positive integer a such that $(0) = (\sqrt{(0)})^a = (m_1 . m_2 . \cdots . m_k)^a = m_1^a, m_2^a, \cdots, m_k^a$. We know that R is a noetherian ring. On the other hand, R is also an artinian ring by lemma 26, § 4.3 in [2]. Hence $\dim R = 0$. This contradicts the hypothesis that $\dim R = n > 2$. The second part of the corollary is thus proved.

From the course of the proof given above, it is easy to see that if R is a noetherian semi-local ring, then R is artinian iff its minimal radical and its maximal radical are identical.

As a consequence of Corollary 1, we have

Corollary 2 If (R, m) is an n-dimensional noetherian local ring and $x \in R - U(R) = m$ ($U(R)$ is the set of all unit of R), then for each i $(0 < i \leqslant n)$, there exists a $P \in \mathrm{Spec}(R)$ such that $h(P) = i$ and $x \in P$.

Corollary 3 If R is an n-dimensional noetherian ring $(n < \infty)$, then for every integer j $(0 < j \leqslant n)$ and each pair of prime ideals S_j, T_j, of R with $h(S_j) = h(T_j) = j$, $S_j \neq T_j$, there exist prime chains $S_1 \subset S_2 \subset \cdots \subset S_j$ and $T_1 \subset T_2 \subset \cdots \subset T_j$ such that $S_i \not\subset T_k, T_i \not\subset S_k$, $1 \leqslant i, k \leqslant j$.

Proof The case that $j = 1$ is obvious, Suppose $j > 1$, there exist $x, y \in R$ with $x \in S_j$, $x \bar{\in} T_j$ and $y \in T_j$, $y \bar{\in} S_j$. By Corollary 1, there exist $S_{j-1}, T_{j-1} \in \mathrm{Spec}(R)$ such that $h(T_{j-1}) = h(S_{j-1}) = j-1$, $S_{j-1} \subset S_j$, $T_{j-1} \subset T_j$ and $x \in S_{j-1}$, $y \in T_{j-1}$. In the same manner, we can obtain the chains $S_1 \subset S_2 \subset \cdots \subset S_j$ and $T_1 \subset T_2 \subset \cdots \subset T_j$ as desired.

Theorem 3 The following conditions of a noetherian ring R are equivalent.

(1) $|\mathrm{Spec}(R)| < \infty$.

(2) R is a semi-local ring and $\dim R \leqslant 1$.

(3) There exists a positive integer k such that for each ideal I of R, the number of prime ideals belonging to I dose not exceed k.

(4) The descending chain of the intersections of prime ideals is stable.

(5) The intersection of some prime ideals which do not mutually contain each other is not a prime ideal.

Proof (1) $\Rightarrow$ (2) by Theorem 2.

Obviously (1) $\Rightarrow$ (3), (4), and (5).

To show (2) $\Rightarrow$ (1). Since noetherian ring R has only a finite number of minimal prime ideals, by the semi-local property, there exists at most a finite number of prime ideals of height 1. This shows $|\mathrm{Spec}(R)| < \infty$.

(3) $\Rightarrow$ (1). Suppose that $|\mathrm{Spec}(R)| = \infty$ and $P_1, P_2, \cdots, P_n, \cdots$ are prime ideals with $h(P_i) = 1$, $i = 1, 2, \cdots$. (This is possible by Theorem 2). Let $I_a = \bigcap_{i=1}^{a} P_i$ then $P_1, P_2, \cdots, P_a$ are minimal ideals belonging to I_a. The arbitrariness of a contradicts the assumption of (3). Thus (1) is a consequence of (3).

(4) $\Rightarrow$ (1). Suppose that $|\mathrm{Spec}(R)| = \infty$ and $P_1, P_2, \cdots, P_n, \cdots$, are prime ideals with $h(P_i) = 1$, $i = 1, 2, \cdots$, If $k > n$ then $\bigcap_{i=1}^{k} P_i \subset \bigcap_{i=1}^{n} p_i$. We claim that $\bigcap_{i=1}^{k} P_i \neq \bigcap_{i=1}^{n} P_i$, for otherwise we will have $P_{n+1} \supset \bigcap_{i=1}^{n} P_i$. Then there exists an integer j $(1 \leqslant j \leqslant n)$ such that $P_{n+1} \supset P_j$ with $h(P_{n+1}) = h(P_j) = 1$. This contradiction implies that $(\bigcap_{i=1}^{a} P_i \mid a = 1, 2, \cdots)$ is not stable. thus (4) implies (1).

(5) $\Rightarrow$ (1). Let $|\mathrm{Spec}(R)| = \infty$, P be a minimal prime ideal and $P_1, P_2, \cdots$,

be prime ideals such that $P_i \supset P$, $h(P_i)=1$, $i=1,2,\cdots$. Let $\bigcap_{i=1}^{\infty} P_i = I$, then $I=P$ because prime ideals belonging to I are finite. This contradicts the condition (5). Hence (1) follows from (5).

As a result of the proof given above, we have

Corollary Let R be a noetherian ring and $\{P_i | i \in V\}$ be an arbitrary infinite set of prime ideals of R which do not mutually contain each other. Then $\bigcap_{i \in V} P_i$ is also a prime ideal.

Reference

[1] Satyanarayana. *M*, Generalized Primary Rings, Math. Ann., 179 (1969) 109.

[2] Feng Keqin, 《Basic Commutative Algebra》(text in Chinese), Higher Education Press, 1985.

Noether 环 R 中的素谱 Spec(R)

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摘 要

本文首先证明了：在二维局部整环（R, m）中，高为 1 的素理想的集 Σ 是无限集，且 $\bigcup_{P \in \Sigma} P = m$，$\bigcap_{P \in \Sigma} P = (0)$．其次给出了当 $\dim R \geqslant 2$ 时 R 中高为 i 的素理想与高为 $i-1$ 或 $i+1$ 的的素理想间的关系：设 R 是 n 维 Noether 环（$n \geqslant 2$，允许 $n=+\infty$），$P_0 \subset P_1 \subset \cdots \subset P_k$ 是 R 中任一极大素链，则 $\forall i\,(0<i<k)$，$\Sigma_i = \{P \in \mathrm{Spec}(R) | P_{i-1} \subset P \subset P_{i+1}\}$ 是无限集，且 $\bigcap_{P \in \Sigma_i} P = P_{i-1}$，$\bigcup_{P \in \Sigma_i} P = P_{i+1}$．特别地，当 n 有限时，选 $k=n$ 知：在 n 维 Noether 环中，高为 $i\,(0<i<n)$ 的素理想必有无限多个．并导出如下几个推论．

1．R 如上，则：

①．$m \in \mathrm{Spec}(R)$，$1 \leqslant h(m) = k$，则 R 中所有含于 m 且高为 $j\,(0<j \leqslant k)$ 的素理想的并等于 m；

②．R 中高为 $i\,(0 \leqslant i<n)$ 的素理想的交等于 $\sqrt{(0)}$，若 $\bigcap_{P \in \max(R)} P = \sqrt{(0)}$，则 R 必不是半局部环．

2．（R, m）是 n 维 Noether 局部环，$x \in R - u(R) = m$，（$u(R)$ 是 R 中所有单位的集），则 $\forall i$，$0<i \leqslant n$，均存在 $P \in \mathrm{Spec}(R)$，$h(P)=i$，$x \in P$．

最后给出了在 Noether 环 R 中，$|\mathrm{Spec}(R)|<\infty$ 的几个等价条件。