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Characteristic Exponents of a Class of Limear Perodic Systems*

Yue Xiting
(Chang Chun Industry School of Building Materials)

Pan Jiaqi
(Northeast Normal University)

1. Introduction

Consider the linear periodic system

$$\dot{X} = A(t)X, \qquad (\cdot = \frac{d}{dt}), \tag{L.P}$$

where $X \in R^n$, $t \in R$, and $A(t) = [a_{ij}(t)]$ is an $n \times n$ real value continuous matrix function of t, satisfying $A(t) = A(t+\omega)$ for some $\omega > 0$.

It is well known that characteristic exponents play important roles in studying system (L.P). But, to determine the Characteristic exponents of (L.P), is an extremely difficult prodlem, so far people only know very little about it, except the scalar second order equations and, more generally, the Hamiltonian and canonical systems, This is because there is no obvious relation between the characteristic exponents and the matrix $A(t)$[1].

In this paper, we consider a class of particular linear periodic system which we call the commutative type of linear periodic system (Definition 1.) and briefly denote as (C.L.P). For the (C.L.P) we explicitly expounded the relation between characteristic exponents and matrix $A(t)$, a simple method of determining the characteristic exponents of the system is given.

Papers [4], [5] have also employod commutative condition and explored the stabilities of general linear system. But, the relation between characteristic exponents and matrix $A(t)$ is exposed in this paper by first time.

2. Primary results

Definition 1 A linear periodic system (I.P) is called a commutative type of linear periodic system (C.I.P), if the cofficient matrix of (I.P) satisfies

$$A(t)A(s) = A(s)A(t) \qquad \text{for all } t, s \in [0, \omega],$$

Lemma 1 The fundamental matrix solution of (C.I.P) is

$$X(t, t_0) = \exp\left[\int_{t_0}^{t} A(s)\,ds\right],$$

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and the solution of (C.L.P) satisfying the initial value problem (t_0, x_0) is

$$x(t, t_0, x_0) = \exp\left[\int_{t_0}^{t} A(s)ds\right] x_0$$

Proof Omitted.

In order to prove our theorem, we generalize the result of [2; p104] to the function matrix.

Lemma 2 If $A(t)$ is an $n\times n$ ω-periodic function matrix,

$$A(t+\omega) = A(t) \quad \text{for some} \quad \omega > 0,$$

then there exists an $n\times n$ scale matrix B and $n\times n$ ω-periodic function matrix $P(t)$ such that

$$\int_{t_0}^{t} A(s)ds = Bt + P(t)$$

holds for all $t\in R$, where $P(t) = P(t+\omega)$, $B = \frac{1}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds$.

Proof If we set $P(t) = \int_{t_0}^{t} A(s)ds - \frac{t}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds$, then $P(t)$ satisfied $P(t+\omega) = P(t)$. In fact,

$$\begin{aligned} P(t+\omega) &= \int_{t_0}^{t+\omega} A(s)ds - \frac{t+\omega}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds \\ &= \int_{t_0}^{t} A(s)ds + \int_{t}^{t+\omega} A(s)ds - \int_{t_0}^{t_0+\omega} A(s)ds - \frac{t}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds \\ &= \int_{t_0}^{t} A(s)ds - \frac{t}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds = P(t). \end{aligned}$$

It follows that,

$$\int_{t_0}^{t} A(s)ds = \int_{t_0}^{t} A(s)ds + \frac{t}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds - \frac{t}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds = Bt + P(t),$$

and the lemma is proved.

Lemma 3 If the periodic function matrix $A(t)$ is commutative, namely $A(t)A(s) = A(s)A(t)$ for all $t, s\in[0,\omega]$, then

$$BP(t) = P(t)B$$

holds for all $t\in R$, where B and $P(t)$ are defined by lemma 2.

Proof In fact,

$$\begin{aligned} BP(t) &= \frac{1}{\omega}\int_{t_0}^{t_0+\omega} A(u)du\cdot\left(\int_{t_0}^{t} A(s)ds - \frac{t}{\omega}\int_{t_0}^{t_0+\omega} A(s)ds\right) \\ &= \frac{1}{\omega}\int_{t_0}^{t_0+\omega} A(u)du\cdot\int_{t_0}^{t} A(s)ds - \frac{t}{\omega^2}\int_{t_0}^{t_0+\omega} A(u)du\cdot\int_{t_0}^{t_0+\omega} A(s)ds \\ &= \frac{1}{\omega}\int_{t_0}^{t}\int_{t_0}^{t_0+\omega} A(u)du\cdot A(s)ds - \frac{t}{\omega^2}\int_{t_0}^{t_0+\omega} A(s)ds\cdot\int_{t_0}^{t_0+\omega} A(u)du \end{aligned}$$

$$=\frac{1}{\omega}\int_{t_0}^{t}A(s)\,ds\cdot\int_{t_0}^{t_0+\omega}A(u)\,du-\frac{t}{\omega^2}\int_{t_0}^{t_0+\omega}A(s)\,ds\cdot\int_{t_0}^{t_0+\omega}A(u)\,du$$

$$=\left(\int_{t_0}^{t}A(s)\,ds-\frac{t}{\omega}\int_{t_0}^{t_0+\omega}A(s)\,ds\right)\cdot\frac{1}{\omega}\int_{t_0}^{t_0+\omega}A(u)\,du=P(t)B.$$

this proves lemma 3.

Theorem Characteristic exponents of the (C. L. P) system are the eigenvalues of the scale matrix B which is given by lemma 2.

Proof We know from lemma 1 that the fundamental matrix solution of (C. L. P) system is

$$X(t,t_0)=\exp\left[\int_{t_0}^{t}A(s)\,ds\right].$$

Also, lemma 2 allows

$$\int_{t_0}^{t}A(s)\,ds=Bt+P(t),$$

where B and $P(t)$ are given by lemma 2. Using the lemma 3, we have

$$X(t,t_0)=\exp\left[\int_{t_0}^{t}A(s)\,ds\right]=\exp[P(t)+Bt]=F(t)\exp[Bt],$$

where $F(t)=\exp[P(t)]$. It is clear that $F(t)=F(t+\omega)$ for all $t\in R$.

This result accords exactly with the Theorem of Floquet-Lyapunov. Therefore, the eigenvalues of the matrix B is the characteristic exponents of (C.L.P) system[3], and the proof of Theorem is completed.

For the convenience of calculating, we also need the following concept.

Definition 2 If the scale function $a(t)$ is ω-periodic function, and satisfies $\int_{t}^{t+\omega}a(s)\,ds=0$ for all $t\in R$, then function $a(t)$ is called the fundamental periodic function.

Corollary 1 Let $A(t)$ be the $n\times n$ cofficient matrix of the (C. L. P) system. If $A(t)$ can be written as $A(t)=A_0+\overline{A}(t)$, where A_0 is a real scale matrix and $\overline{A}(t)$ is composed of fundamental periodic functions, then the eigenvalues of A_0 are the characteristic exponents of (C. L. P) system.

Proof It is obvious that $A_0=B$, where B is given by lemma 2.

If the matrices B and A_0 are given by lemma 2 and corollary 1 respectively, then we also have the following conclusion.

Corollary 2 The trivial solution of (C. L. P) system is uniformly asymptotically stable if and only if all eigenvalues of matrix B (A_0) have negative real parts.

In particular, if this is the case and $X(t)$ is a matrix solution of the (C.L.P) system, then there exist constants $k>0, a>0$ such that

$$|X(t)X^{-1}(s)| \leqslant k\exp[-a(t-s)] \quad \text{for} \quad t \geqslant s.$$

The proof of corollary 2 is referred to Chapter 3 of [1].

3. Example

Let us consider the 2×2 (C.L.P) system

$$\dot{X} = A(t)X \tag{1}$$

where

$$A(t) = \begin{bmatrix} -1 + a\cos\omega t\cdot\sin\omega t & a\cos^2 t + \omega \\ -a\sin^2\omega t - \omega & -1 - a\cos\omega t\cdot\sin\omega t \end{bmatrix}$$

$X = (x_1, x_2)^T$, a, ω are constants.

The coefficient matrix $A(t)$ satisfied the commutative condition and can be written as $A(t) = A_0 + \overline{A}(t)$, where

$$A_0 = \begin{bmatrix} -1 & \omega + \frac{a}{2} \\ -\omega - \frac{a}{2} & -1 \end{bmatrix} \qquad \overline{A}(t) = \begin{bmatrix} \frac{a}{2}\sin 2\omega t & \frac{a}{2}\cos 2\omega t \\ \frac{a}{2}\cos 2\omega t & -\frac{a}{2}\sin 2\omega t \end{bmatrix}$$

The matrix $\overline{A}(t)$ is composed of fundamental periodic function. By corollary 1, the eigenvalues of matrix A_0 are the characteristic exponents of system (1), and the eigenvalues of A_0 are

$$\lambda_{1,2} = -1 \pm i\left(\frac{a+2\omega}{2}\right).$$

From corollary 2, it follows that the trivial solution of system (1) is exponentially asymptotically stable.

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一类线性周期系统的特征指数

岳锡亭　　潘家齐

摘　要

众所周知，在常微分方程理论研究中，线性周期系统的特征指数在该系统的研究中起着极其重要的作用。然而如何确定特征指数，是一个极为困难的工作，除了某些具体方程，至今所知甚少，其原因是人们一直认为特征指数与系统的系数阵之间没有直接关系。

本文考虑了一类特殊的线性周期系统，我们称之为可换型线性周期系统，即系数阵满足某种可换条件。对于这种系统，我们利用其“平均”系统，找到系统的特征指数与其系数阵之间的直接关系，从而给出了确定该线性周期系统特征指数的一种简单方法。