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A Discussion on Some Problems of Lyapunov Matrix Equation about the Partial Stadility*

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Consider Lyapunov matrix equation

$$A^T B + BA = -C \qquad (1)$$

and the linear systems with constant coefficient

$$\dot{x} = Ax \qquad (2)$$

where $A, B, C \in R^{n\times n}$, $B^T = B$, $C^T = C$.

A well-known result is the following.

Theorem (Lyapunov). The equilibrium of the systems (2) is asymptotically stable if and only if for an arbitrarily given positive definite matrix C there exists a positive definite solution matrix B of $Eq.$(1).

Recently the papers [1,2,3,4] study the relationship between $Eq.$(1) and the partial stability of the equilibrium of the systems (2). This paper is a commentary on aforesaid works. In this paper we discuss several problems and point out some mistakes.

I. On partial positive definite quadratic form.

The notion of partial positive definite function was first defined by Rumyantsev (paper [5]). Of course, if a function is positive definite, then it is also partial positive definite.

A partial positive definite quadratic form $f(x_1, \cdots, x_n)$ for $x_1, \cdots, x_m$ is said to be strict partial positive definite, if it is not positive definite for variables $x_1, \cdots, x_m$, $x_i\,(i = m+1, \cdots, n)$. For example, $f(x_1, x_2, x_3) = x_1^2 + (x_1 + x_2 + x_3)^2$ is strict positive definite for x_1; $f(x_1, x_2, x_3) = x_1^2 + (x_1 + x_2)^2$ is positive definite for x_1, but not strict. It is clear that any positive definite function is not strict partial positive definite. The formulation of the notion on partial positive definite function can avert some confusions.

For example, when matrix C is singular, the solution matrix of $Eq.$(1) may be strict partial positive definite and not positive definite, unless the equilibrium

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of the systems (2) is stable.

The proposition in paper [4] points out that a quadratic form x^TCx with rank $C=r$ is positive definite for $x_1, \cdots, x_m$ if and only if

$$x^TCx=(p_{11}x_1+\cdots+p_{1m}x_m)^2+\cdots+(p_{m1}+\cdots+p_{mm}x_m)^2+(p_{m+1,1}x_1+\cdots+p_{m+1,n}x_n)^2+\cdots+(p_{r1}x_1+\cdots+p_{rn}x_n)^2 \quad (3)$$

where the matrix $(p_{ij})_{m\times m}$ is nonsingular. By formula (3) we can easily determine whether a partial positive definite quadratic form is strict or not.

2 . On the matrix C in equation (1).

First, solving matrix equation (1) means that for some kind of partial positive definite matrix C and a given matrix A there exists a partial definite solution matrix B of *Eq*. (1). Thus the study in paper [2] is not solving *Eq*. (1).

Secondly, paper [1,3,4] all obtained a sufficient condition of solvability of *Eq*. (1), i.e., rank $C=m$. But paper [4] also points out that rank $C=m$ is not necessary to the solvability of *Eq*. (1). If rank $C>m$, there is either solvable or unsolvable case of *Eq*. (1).

Thirdly, rank $(A^TB+BA)=m+h$ in paper [2] is wrong.

Example I 1)

$$A=\begin{bmatrix}-5 & 1 & 1\\ -8 & 2 & 1\\ -8 & 1 & 2\end{bmatrix}$$

The equilibrium of $\dot{x}=Ax$ is asymptotically stable for x_1, and $m=1$, $h=1$.

For $B=\frac{1}{4}\begin{bmatrix}10 & -3 & -3\\ -3 & 1 & 1\\ -3 & 1 & 1\end{bmatrix}$, we have $A^TB+BA=\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$, rank $(A^TB+BA)=1<m+h$.

2)

$$A=\begin{bmatrix}-1 & 0 & 0\\ -1 & 1 & 2\\ 3 & -1 & -2\end{bmatrix}.$$

The equilibrium of $\dot{x}=Ax$ is asymptotically stable for x_1 and $m=1$, $h=0$.

For $B=\frac{1}{4}\begin{bmatrix}27 & 5 & 10\\ 5 & 2 & 4\\ 10 & 4 & 8\end{bmatrix}$, we have $A^TB+BA=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & -2\\ 0 & -2 & -4\end{bmatrix}$, rank$(A^TB+BA)=2>m+h$.

3 . On the positive definiteness and the uniqueness of solution matrix B.

Let the equilibrium of systems (2) be asymptotically stable for $x_1, \cdots, x_m$ and the quadratic form x^TCx be positive definite for $x_1, \cdots, x_m$ with rank $C=m$, then there generally exists an infinite solution set of *Eq*. (1). But solution matrices B are not all partial positive definite (see paper [4]).

And in paper [3] the uniqueness of partial positive definite solution matrix

B also does not hold. Actually, there are some mistakes about the proof of Theorem 2.1 in paper [3].

Example 2

$$A=\begin{bmatrix} -1 & 0 & 0 \\ -1 & 2 & 1 \\ 3 & -2 & -1 \end{bmatrix}$$

The equilibrium of systems (2) is asymptotically stable for x_1. Let $C=\mathrm{diag}(1,0,0)$ we can obtain solutions of *Eq*. (1) as follows.

$$B=\begin{bmatrix} b_1+2b_2+1/2 & 2b_1-2b_2 & b_1 \\ 2b_1-2b_2 & b_2 & b_2 \\ b_1 & b_2 & b_2 \end{bmatrix}$$

where b_1、b_2 are any real numbers.

B is positive semidefinite if and only if $b_2 \geqslant 0$, $b_1=2b_2$. Thus we have

$$B=\begin{bmatrix} 4b_2+1/2 & 2b_2 & 2b_2 \\ 2b_2 & b_2 & b_2 \\ 2b_2 & b_2 & b_2 \end{bmatrix}$$

and quadratic form $x^TBx=\frac{1}{2}x_1^2+b_2(2x_1+x_2+x_3)^2$. $(b_2\geqslant 0)$. It is just strict positive definite for x_1. Hence, there exists an innumerable partial positive definite solution matrix of *Eq*. (1).

References

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关于部分变元Lyapunov矩阵方程几个问题的讨论

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摘　　要

关于Lyapunov矩阵方程 $A^TB+BA=-C$的解与线性定常系统 $\dot{x}=Ax$之零解的部分变元渐近稳定性的关系，本文就最近发表的一些结果讨论了如下几个问题．一、由于全变元正定函数也满足部分正定性的条件，有必要引进严格部分正定函数的定义．严格部分正定函数与全变元正定函数是互不相包的；二、求解矩阵方程即意味着对于给定的矩连A(它使系统$\dot{x}=Ax$之零解对部分变元渐近稳定)，矩阵C应满足什么条件使矩阵方程有解B，此即 Lyapunov函数的存在与构造问题；本文还指出C的秩不必为m，当 rank$C>m$时矩阵方程可能有解或无解；又(A^TB+BA)的秩不一定等于$m+h$；三、对于满足有关定理条件的矩阵A、C，矩阵方程 $A^TB+BA=-C$的解B中不仅有部分正定的，亦可能有正定的、不定的等，而其中部分正定的解矩阵B的唯一性并不成立．

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6） $\min\{F^2(a_{2r+1}),\ F^2(\overline{a}_{2r+1})\}\geqslant[(K_{2r}M_{2r}-m_{F,2r})^2+M^2_{G,2r}]$.

7） $F(a_{2r})>K_{2r}M_{2r-1}$, $F^2(\overline{a}_{2r+1})<-K_{2r}M_{2r-1}$.

where 4), 6), 7) are holding for $r=1,2,\cdots,k$.

In this paper, at first we construct eight lemmas, secondly we deduce six estimating expressions for the values of the path-curves of the Lienard equation, last we obtain the following result.

Theorem If the Lienard equation, satisfies the conditions from 1) to 7) as stated above, then it has $n=2k$ limit-cycles at least. If it satisfies the conditions from 1) to 7), and 7) is holding for $r=k+1$; then it has $n=2k+1$ limit-cycles at least.

Example Let $f(x)=-(x^2-1)(x^2-3^2)$ and $g(x)=\frac{1}{2}x$ in Lienard equation, then it has two limit-cycles on the interval $[-5,5]$ at least.