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On the Existence and Uniqueness of Connecting Orbits for two and Three Dimensional Cooperative Systems*

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I. Introduction

Many physical problems have been represented by systems of autonomous ordinary differential equations, in which the solution to the physical problem is given by an orbit connecting two critical points. At the same time, much theoretical research has been concerned with the global structure of dynamical systems. In recent years, a considerable number of papers have been written in connection with this subject (see [1—5]).

Consider a C^1 system of differential equation defined in N

$$\dot{x}=F(x) \tag{1}$$

where $N\subset \mathbf{R}^n$ is an open set. The system (1) is called cooperative if $\frac{\partial F_i}{\partial x_j}\geqslant 0$ for all $i\neq j$ and for all $x\in N$. We assume that the system (1) is cooperative and has critical points $O=(0,0,\cdots,0)$ and $Q=(q_1,q_2,\cdots,q_n)$, where $q_i>0$ for $i=1,2,\cdots,n$. Conlon [4] has proved that if F is C^2 and each off-diagonal term of $DF(x)$ is positive, then there is a unique orbit of F contained in $B(O,Q)=\{x\in \mathbf{R}^n: 0<x_i<q_i$ for $i=1,2,\cdots,n\}$ which joins O and Q. This result is a generalization of a 2-dimensional result of Conley and Smoller [1]. Selgrade [5] has given an n-dimensional result whith is the same as that of Conlon [4] except he replaced C^2 vector field with a C^1 vector field and he weakened the positivity assumption on the off-diagonal terms of $DF(x)$. All authors mentioned above have studied this problem under the assumption that the principal eigenvalues of $DF(O)$ and $DF(Q)$ are nonzero. When one of these two critical points is degenerate, i.e., one of these principal eigenvalues is zero, the problem for the existence and uniqueness of orbits connecting two critical points has not been solved yet.

The purpose of the present paper is to study this problem for two and three

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dimensional cooperative systems. We shall give two theorems about the existence and uniqueness of orbits connecting two critical points, which contain the degenerate case. The tool made use of is the monotonicity theory and Ważewski retraction method.

2. The Main Results

First of all, we give the following notation and definition.

Let $x, y \in \mathbf{R}^n$. There is a partial order on $\mathbf{R}^n$ given by $x \leqslant y$ $(x < y)$ if and only if $x_i \leqslant y_i$ $(x_i < y_i)$ for $i = 1, 2, \cdots, n$. Assume that N is a neighborhood of $B(0, Q)$ and $F: N \to \mathbf{R}^n$ is a C^1 map. If the system (1) is cooperative, then by the Kamke theorem the flow $\{\phi_t\}_{t>0}$ generated by (1) has the property: if $x \leqslant y$, then $\phi_t(x) \leqslant \phi_t(y)$ for $t \geqslant 0$.

The system (1) is called irreducible, if $DF(x)$ is irreducible for all $x \in N$, that is, $DF(x)$ leaves invariant no nontrivial coordinate subspaces of $\mathbf{R}^n$.

Set

$$[p, q] = \{x \in \mathbf{R}^n : p \leqslant x \leqslant q\}, \quad [[p, q]] = \{x \in \mathbf{R}^n : p < x < q\}.$$

In this paper, we shall prove the following theorems

Theorem 1 Suppose $n = 2$, and $F: N \to \mathbf{R}^2$ is a C^1 map. If the system (1) is cooperative in $B(0, Q)$ with critical points only at O and Q, then there is an orbit contained in $B(0, Q)$ which joins O and Q. Furthermore, if $DF(O)$ and $DF(Q)$ are irreducible, then this connecting orbit is unique in $B(O, Q)$.

Theorem 2 Suppose $n = 3$, and $F: N \to \mathbf{R}^3$ is a C^1 map. If the system (1) is cooperative and irreducible in $B(O, Q)$ with critical points only at O and Q, then there is a unique orbit of F contained in $B(O, Q)$ which joins O and Q.

3. Proof of the Theorems

Proof of Theorem 1 Let L denote the open line segment with endpoints O and Q. By Theorem 2.3 in [6], for any $p \in [O, Q]$, either $\lim_{t\to\infty} \phi_t(p) = 0$, or else $\lim_{t\to\infty} \phi_t(p) = Q$. Choose $q \in L$, without loss of generality, we assume that $\lim_{t\to\infty} \phi_t(q) = Q$. From the Kamke theorem it follows that $\lim_{t\to\infty} \phi_t(p) = Q$ for $p \in [q, Q]$. We claim that $\lim_{t\to\infty} \phi_t(p) = Q$ for any $p \in L$. If not, there is $p_0 \in L$ such that $\lim_{t\to\infty} \phi_t(p_0) = 0$. Therefore, $\lim_{t\to\infty} \phi_t(p) = 0$ for any $p \in [O, p_0]$. Set $q_0 = \sup\{p \in L : \lim_{t\to\infty} \phi_t(p) = 0\}$. Then $0 < q_0 < Q$. Since $\phi_t(q)$ is continuous in q, $\lim_{t\to\infty} \phi_t(q_0) = Q$.

Set

$$S(O) = \{p \in [O, Q] : \lim_{t\to\infty} \phi_t(p) = O\},$$

$$S(Q)=\{p\in[O,\ Q]:\lim_{t\to\infty}\phi_t(p)=Q\}.$$

It is easy to see that $S(O)$ and $S(Q)$ are two nonempty open sets of $[O, Q]$ which are disjoint. By $q_0\in S(Q)$, we conclude that $S(O)\cap S(Q)\neq\phi$. This contradiction shows the claim.

Set

$$l_1=\{(0,y):\ 0<y<q_2\},\qquad l_2=\{(x,q_2):\ 0<x<q_1\},$$
$$l_3=\{(x,0):\ 0<x<q_1\},\qquad l_4=\{(q_1,y):0<y<q_2\}.$$

We assert that there exists $p_i\in l_i$ such that $\phi_t(p_i)\bar{\in}[O,Q]$ for $t\in(-\varepsilon,0)$ where $\varepsilon>0$ is sufficiently small and $i=2,4$. In fact, if there is $p_2\in l_2$ such that $F_2(p_2)\neq 0$, hence $F_2(p_2)<0$. Such a point p_2 satisfies the condition we need; If for any $p\in l_2$, $F_2(p)=0$, then $p_2=(0,q_2)$ is the point we need. The existence of p_4 can be proved by similar way.

Let Γ be the closed line segment with endpoints p_2 and p_4. Set $A=\{p\in\Gamma$: there is $t(p)\leq 0$ such that $\phi_{t(p)}(p)\in l_1\cup l_2$ and $\phi_t(p)\bar{\in}[O,Q]$ for $t\in(t(p)-\varepsilon,t(p))$, where $\varepsilon>0$ is sufficiently small$\}$, $B=\{p\in\Gamma$: there is $t(p)\leq 0$ such that $\phi_{t(p)}(p)\in l_3\cup l_4$ and $\phi_t(p)\bar{\in}[O,Q]$ for $t\in(t(p)-\varepsilon,t(p))$, where $\varepsilon<0$ is sufficiently small$\}$.

By continuity of ϕ and definition of A and B, A and B are open sets of Γ which are disjoint. It follows from the connectivity of Γ that $A\cup B\neq\Gamma$, i.e., there is a point $p\bar{\in}\Gamma-(A\cup B)$ such that $\phi_t(\bar{p})\in[O,Q]$ for any $t<0$. So $\lim_{t\to-\infty}\phi_t(\bar{p})=0$ or Q. By definition $\bar{p}\neq p_i$ for $i=2,4$, hence, $0<\bar{p}<Q$. As proved above, $\bar{p}\in S(Q)$. It is easy to prove that $\lim_{t\to-\infty}\phi_t(\bar{p})=0$, that is, $\phi_t(\bar{p})$ joins O and Q. If $DF(O)$ and $DF(Q)$ are irreducible, then proof of uniqueness of such a connecting orbit is similar to that of Selgrade [5]. The proof of Theorem 1 is complete.

In order to prove Theorem 2, we first show the following lemma.

Lemma 1 Either $L\subset S(O)$ or else $L\subset S(Q)$ (these notations are the same as those in proof of Theorem 1).

Proof The ω-limit set and α-limit set of p are denoted, respectively, by $\omega(p)$ and $\alpha(p)$. Set $M=\{p\in[O,Q]:\omega(p)\not\subset\{O,Q\}\}$. By Theorem 4. in [7], M has measure zero. Therefore, there exists $p\in[[O,Q]]$ such that either $\omega(p)=0$ or $\omega(p)=Q$. Without loss of generality, we assume that $\omega(p)=Q$. By the Kamke theorem, $[p,Q]\subset S(Q)$. To complete the proof of this lemma, it suffices to prove that $L\subset S(Q)$. Suppose not, there exists $p_0\in L$ such that $\omega(p_0)\neq Q$. From the Kamke theorem it follows that there is no point $p\in[0,p_0]$ such that $\lim_{t\to\infty}\phi_t(p)=Q$. Since $M\cap[0,p_0]$ has measure zero, there is $p_1\in[[0,p_0]]$ such that $\lim_{t\to\infty}\phi_t(p_1)=0$. By the Kamke theorem, $[0,p_1]\subset S(O)$. Set

$$q_0=\inf\{q\in L: \omega(q)=Q\}, \qquad q_0'=\sup\{q\in L: \omega(q)=0\}.$$

By continuity of $\phi_t(p)$, $q_0\bar{\in} S(Q)$ and $q_0'\bar{\in} S(O)$. Since M has measure zero, $q_0=q_0'$. It is easy to see that $\omega(q_0)\cap(O, Q\}=\phi$. From Theorem 4.1 in [6], we know that $\omega(q_0)$ is a closed orbit, and $0<\omega(q_0)<Q$. Since $\omega(q_0)$ is compact, there are $q_i\in L$ for $i=1,2$ such that $[O, q_1]<\omega(q_0)<[q_2, Q]$. Set

$$K=\{x\in \boldsymbol{R}^3: x \text{ is not related to any } y\in\omega(q_0) \text{ by} <\}.$$

From Theorem 2.4 in [8], K contains a bounded and connected subset $K(y)$ which is negatively invariant such that $K(y)\subset[O, Q]$. Moreover $K(y)$ contains at least one critical point u of (1), where $y=\omega(q_0)$. By definition, $[O, q_1]\cap K(y)=\phi$, and $[q_2, Q]\cap K(y)=\phi$. Therefore, $u\in[O, Q]-\{O, Q\}$, which contradicts ths assumption of Theorem 2. This completes the proof.

Proof of Theorem 2 By strong monotonicity of $\phi_t(p)$, (see [7, Thm 5.1]) and Lemma 1, we can suppose that $[O, Q]-\{O\}\subset S(Q)$, and all points in $\overline{N}=Bd([O, Q])-\{O, Q\}$ are strict ingress point, where Bd denotes the boundary of a set.

Let π_1 denote the plane passing through $(0,0,q_3)$, $(q_1/2,0,0)$ and $(0,q_2/2,0)$, and let π_2 denote the plane passing through (q_1,q_2,q_3), $(q_1,q_2/2,0)$ and $(q_1/2, q_2,0)$. Set

$$\Sigma_i=[O, Q]\cap\pi_i \text{ for } i=1,2,$$
$$S^2=\{(x,y,z)\in\boldsymbol{R}^3: x^2+y^2+z^2=1\},$$
$$B=\{(x,y,z)\in\boldsymbol{R}^3: x^2+y^2+z^2\leqslant 1\},$$
$$S_*^2=S^2-\{(0,0,\pm 1)\}.$$

Since $\overline{N}$ is homeomorphic to S_*^2, $\overline{N}$ is not contracible. We can easily prove that $\overline{N}\cup\Sigma_1\cup\Sigma_2$ is homeomorphic to S^2. Therefore, $\overline{N}\cup\Sigma_1\cup\Sigma_2$ is contracible. Then $\overline{N}$ is not a retract of $\overline{N}\cup\Sigma_1\cup\Sigma_2$.

We claim that there is a point $p_0\in\Sigma_1\cup\Sigma_2$ such that $\phi_t(p_0)\in[[O, Q]]$ for any $t\leqslant 0$. If not, for any $p\in\Sigma_1\cup\Sigma_2$, there is a number $t(p)\leqslant 0$ such that $\phi_{t(p)}(p)\in\overline{N}$, where $t(p)=\inf\{t\leqslant 0:\phi_\sigma(p)\in[O, Q] \text{ for } \sigma\in[t,0]\}$.

Let the mapping $f:\overline{N}\cup\Sigma_1\cup\Sigma_2\to\overline{N}$

$$f(p)=\begin{cases}\phi_{t(p)}(p) & \text{if} \quad p\in\Sigma_1\cup\Sigma_2,\\ p & \text{if} \quad p\in\overline{N}.\end{cases}$$

It is easy to prove that f is continuous, that is, $\overline{N}$ is a retract of $\overline{N}\cup\Sigma_1\cup\Sigma_2$, which is impossible, according to the fact proved above. This shows that there is a point $p_0\in\Sigma_1\cup\Sigma_2$ such that $\phi_t(p_0)\in[O, Q]$ for $t\leqslant 0$. Therefore, $a(p_0)\subset[O, Q]$.

As proved above, $a(p_0)\cap\{O, Q\}=\phi$. It follows from Theorem 2.2 in [6] that no two ponts of $a(p_0)$ can be related by $<$. Hence, $a(p_0)$ only contains one point. It follows from Theorem 2.2 in [7] that $a(p_0)=0$, i.e., $\phi_t(p_0)$ joins O and Q. The proof of uniqueness of such a connecting orbit is similar to that of Selgrade [5].

Remark In the proof of Theorem 2, Lemma 1 plays an important role , After completing this paper, we see Hirsch's paper [9]. He has given an n-dimensional result about lemma 1 (see [9, Thm 10.5]). However, the method used here is not valid for $n>4$. The reason is that N is contracible for $n>4$.

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二维和三维合作系统连结轨线的存在和唯一性

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摘 要

在本文中，我们给出了两维和三维合作向量场的给定两个有序奇点间的连接轨线的存在和唯一性定理，其结果包含了奇点是退化的情形.