Journal of Mathematical Research and Exposition
Vol.10, No.2, May, 1990

Several Proximinal Problems In Banach Spaces*

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1. Introduction

Let X be a Banach space, and U, V two subspaces in X. We mainly consider the proximinality of $U+V$ in X. Other papers directly related to this work are [1], [2]. We prove that if U, V are two proximinal subspaces, V is reflexive, and $U+V$ is closed, then $U+V$ is proximinal in X. In [1], the similar theorem requires that $U\cap V$ is finite dimesional. In section 3, We prove that the proximinality in L^1 and L^p is related.

Let X be a Banach space, G a subspace of $X, x\in X$, $g\in G$, g is said to be a best approximant of x with respect to G, if

$$\|x-g\| = \text{dist}(x,G) = \inf_{x\in G}\|x-g\|$$

G is said to be a proximinal set if and only if for any $x\in X$, there exists at least one best approximant in G.

2. Basic results

We need several results which are not in the mainstream of our arguement. For convenience, we collect them in this section.

Lemma 1. [2] Suppose that U, V are two closed subspaces in Banach space X, then $U+V$ is closed if and only if there exists a constant $K>0$, such that each element $x\in X$ has a representation $x=u+v$ with $u\in U$, $v\in V$ and

$$\max\{\|u\|, \ \|v\|\}\leqslant K\|x\|$$

Lemma 2. [5] If X is a Banach space, X_0 is a finite dimensional or codimensional subspace, then there exists an $X_1\subset X$ such that $X=X_0+X_1$ is topologically direct sum.

Lemma 3. [3] Let U, V be two subspaces in normed space X. If U is proximinal and each element $x\in X$ has a weakly compact subset $K(x)\subset V$ with the property

$$\inf_{x\in K(x)}\text{dist}(x-v,U) = \inf_{v\in V}\text{dist}(x-v,U).$$

*Received Nov.12, 1988.

Then $U+V$ is proximinal.

We make use of above lemmas to prove following theorems.

Theorem 1 Let U, V be two closed subspaces in Banach space X. If $U\cap V$ is a finite or proximinal co-dimensional subspace, then $U+V$ is proximinal if and only if each $f\in X$ responds $u\in U$, $v\in V$ such that

$$\text{dist}(f-u-v, U\cap V)=\text{dist}(f, U+V)$$

Proof Let $f\in X$, then there exist $u\in U, v\in V$ such that

$$\text{dist}(f-u-v, U\cap V)=\text{dist}(f, U+V)$$

Since $U\cap V$ is finite dimensional or proximinal, co-dimensional then there exists a $g\in U\cap V\subset U$ such that

$$\|f-u-v-g\|=\text{dist}(f-u-v, U\cap V)=\text{dist}(f, U+V)$$

i.e. $u+g+v$ is a best approximant of f's.

Necessity: Firstly, let $U\cap V=\{0\}$. Since $U+V$ is proximinal, then there exist $u\in U, v\in V$ such that

$$\|f-u-v\|=\text{dist}(f, U+V)$$

that is

$$\text{dist}(f-u-v, U\cap V)=\text{dist}(f, U+V)$$

Secondly, suppose $U\cap V\neq\{0\}$. From assumaption and lemma 2, we know that there exists a $V_1\subset V$ with $U+V=U+V_1$, furthermore $U\cap V_1=\{0\}$, so we get

$$\|f-u-v\|=\text{dist}(f-u-v, U\cap V_1)\geqslant\text{dist}(f-u-v, U\cap V)$$

but $U\cap V$ is proximinal, then there exists a $u'\in U\cap V$ such that

$$\|f-u-v\|\geqslant\|f-u-v-u'\|=\text{dist}(f-u-v, U\cap V)$$

Also

$$\|f-(u+u')-v\|\geqslant\text{dist}(f, U+V)=\text{dist}(f, U+V_1)$$

From above two inequqlities, we get

$$\text{dist}(f-u-v, U\cap V)=\text{dist}(f, U+V) \qquad \blacksquare$$

Theorem 2 Let U, V be closed subspaces in Banach space X. If U is proximinal, V reflexive and $U+V$ closed, then $U+V$ is proximinal.

Proof. Let A be a proximity projection: $A: X\longrightarrow V$, i.e.

$$\|x-Ax\|=\text{dist}(x, V)$$

We define a projection $B: X\longrightarrow V$, $By=A(f-y)$, where f, y are two elements in X. We will prove that B is a weakly compact operator. Since Projection A is a proximity projection, we get

$$\|f-y-A(f-y)\|=\text{dist}(f-y, V)$$

Also $$\|A(f-y)\|\leqslant\|f-y\|+\text{dist}(f-y, V)$$

If y is bounded, so are $A(f-y)$ and By. we know a Banach space is reflexive if and only if each bounded closed subset is weakly compact. So B projects each bounded subset of X into relatively weakly compact one. furthermore B is a weakly compact operator.

Suppose $f \in X$, then there exists $\{q_n\} \subset U+V$ such that

$$\|f-q_n\| \longrightarrow \text{dist}(f, U+V)$$

From lemma 1, we know there exist $f_n \in U, g_n \in V$ and a constant $K \geqslant 0$ such that $q_n = f_n + g_n$, $\max\{\|f_n\|, \|g_n\|\} \leqslant K\|q_n\|$, where q_n is bounded, so are f_n and g_n. Let $g'_n = A(f-f_n)$, from above proof, we get that $\{g'_n\}$ is relatively weakly compact. Suppose that $\{\bar{g}'_n\}$ is $\{g'_n\}'$s weak closure, we get

$$\inf_{g' \in \{\bar{g}'_n\}} \text{dist}(f-g', U) \leqslant \text{dist}(f-g', U) \leqslant \|f-g'_n-f_n\| \leqslant \|f-g_n-f_n\|$$

So $\inf_{g' \in \{\bar{g}'_n\}} \text{dist}(f-g', U) \leqslant \text{dist}(f, U+V) = \inf_{g \in V} \text{dist}(f-g, U)$

From lemma 3, we get $U+V$ is proximinal. ■

A closed subset G in normed space is said to be very-non-proximinal, if there exists no x in $X \setminus G$ which has best approximant in G.

Theorem 3: Let H be very-non-proximinal subpace, G any subspace. If $G \cap H = \{0\}$ $G+H \neq X$, and $G+H$ is closed, then $G+H$ is very-non-proximinal.

Proof: Suppose the theorem is not true, then there exists an $f \notin G+H$, such that f has best approximant in $G+H$, denoted as $g+h$, where $g \in G, h \in H$, such that

$$\|f-g-h\| = \text{dist}(f, G+H)$$

Since $G \cap H = \{0\}$, then $f-g \notin H$. In fact, if $f-g \in H$, then there exists $f \in g+H \subset G+H$ that contradicts to $f \notin G+H$, so $f-g \notin H$.

Since H is a very-non-proximinal subspace, then

dist $(f-g, H) < \|f-g-h'\|$ for any $h' \in H$, especially when h supposed above.

Then

$$\text{dist}(f, G+H) = \|f-g-h\| > \text{dist}(f-g, H)$$

But $g+h$ is a best approximant of f's in $G+H$, and for any $h' \in H$

$$\|f-g-h'\| \geqslant \|f-g-h\| \quad \text{dist}(f, G+H)$$

This leads a contradiction. ■

3. Two Special Results

Let $C(S, Y)$ be the Banach space of continuous mappings from S to Y. $C(S, Y)$ is endowed with the supremum norm

$$\|f\| = \sup\{\|f(s)\|_Y, s \in S\}.$$

Let T be a vector measure space, Y is Banach space, $L^p(T, Y)$ $(1 \leqslant p < \infty)$ is Banach space of L^p integrable mapping, i.e. $f: T \longrightarrow Y$

$$\|f\| = (\int_T \|f(t)\|^p \text{dm}(t))^{1/p} < \infty.$$

Theorem A: [2] Let T be a compact Hausdorff space, and V is subspace, of Banach space Y, if there exists a continuous proximity $A: Y \longrightarrow V$, then $C(T, V)$ is proximinal in $C(T, Y)$.

The theorem A give a sufficient condition in order that $C(T,V)$ is proximinal. We will give a necessary condition.

Theorem 4: Let T be a compact Hausdorff space and V a non-proximinal subspace of Banach space Y. Then $C(T,V)$ is non-proximinal in $C(T,Y)$.

Proof: Suppose that $C(T,V)$ is proximinal, then for any $f\in C(T,Y)$, there exists at least one $g\in C(T,V)$ such that

$$\|f-g\|=\operatorname{dist}(f,C(T,V))$$

From [3], we get

$$\operatorname{dist}(f,C(T,V))=\sup_{t\in T}\operatorname{dist}(f(t),V)$$

Let $d(y)=\operatorname{dist}(y,V)$, then $d(y)$ is continuous with respect to y, and f is continuous, then $d(t)=d(f(t))=\operatorname{dist}(f(t),Y)$ is continuous.
V is non-proximinal in Y, i.e. there exists an element $x_1\in Y$, such that it has no best approximant in V. So there exists no y in Y such that

$$\|x_1-y\|=\operatorname{dist}(x_1,V)$$

Suppose $f(t)=x_1$, then f has one best approximant g in $C(T,V)$ such that

$$\|f-g\|=\sup_{t\in T}\operatorname{dist}(f(t),V)$$

Since T is compact, then $\sup_{t\in T}\operatorname{dist}(f(t),V)$ can be reached, i.e. there exists t_1 in T such that $\sup_{t\in T}\operatorname{dist}(f(t),V)=\operatorname{dist}(f(t_1),V)$

$$\|f-g\|=\max_{t\in T}\|f(t)-g(t)\|=\max_{t\in T}\|x-g(t)\|$$

where g is continuous, then there exists a $t'\in T$ such that

$$\|f-g\|=\|x-g(t')\|$$

That is

$$\|f-g\|=\|x-g(t')\|=\operatorname{dist}(f(t_1),V)=\operatorname{dist}(x,V)$$

i.e. $g(t')$ is one best approximant of $x's$. This leads a contradiction.

Lemma: [4] Let Y be a closed proximinal subspace of X. Then every simple function $z=\sum_{i=1}^{n}l_{A_i}\otimes v_i\ (A_i\cap A_j=\Phi$ if $i\neq j)$ in $L^p(T,Y)$ has best approximant in $L^p(T,X)$ $(1\leqslant p<\infty)$.

Theorem 5: Let T be a measurable space with finite measure, and Y is a reflexive subspace of Banach space X. If $f\in L^p(T,X)$ $(1\leqslant p<\infty)$, then f has same best approximant in $L^1(T,Y)$ (in L^1-norm) and $L^p(T,Y)$ (in L^p-norm).

Proof: $f\in L^p(T,X)$, then there exists simple function sequence $\{g_n\}\subset L^p(T,X)$ such that $\|g_n-f\|\to 0$ i.e. $(\int_T\|g_n(t)-f(t)\|^p dm(t))^{1/p}\to 0$
By Holder inequality:

$$\int_T\|g_n(t)-f(t)\|dm(t)\leqslant(\int_T\|g_n(t)-f(t)\|^p dm(t))^{1/p}(m(T))^{1/q}\to 0$$

i.e. $g_n(t)\longrightarrow f(t)$ in L^1-norm.

Suppose simple function $g_n=\sum_{i=1}^{m} l_{A_i}\otimes v_i^{(n)}\ (A_i\cap A_j=\Phi\ \ i\neq j)$.

By above lemma $\overline{g_n}=\sum_{i=1}^{m} l_{A_i}\otimes \bar{v}_i^{(n)}$ is a best approximant of $g'_n s$.

where $\overline{v_i}^{(n)}$ is a best approximant of $v_i^{(n)}s$ in Y.

Set $G=\{g_1,g_2\cdots\cdots g_n\cdots\cdots f\}$ is compact in $L^p(T,X)$ $(1\leqslant p<\infty)$.

Set $\overline{G}=\{\overline{g_1},\overline{g_2}\cdots\cdots\overline{g_n}\cdots\cdots\}$ then $\overline{G}$ is relatively, weakly compact subspace in $L^1(T,X)$.

In fact, since Y is a reflexive subspace of X, by Dunford [3], we only prove G is bounded and uniformly integrable.

$$\|g_n\|_1\leqslant\|\overline{g_n}-g_n\|_1+\|g_n\|_1\leqslant 2\|g_n\|_1.$$

Since G is compact, then $\{g_n\}$ is uniformly integrable, so when $\mathrm{mes}(A)\to 0$

$$\int_A\|\bar{g}_n(t)\|\mathrm{dm}(t)\leqslant 2\int_A\|g_n(t)\|\mathrm{dm}(t)\to 0$$

i.e. G is uniformly integrable. So G is relatively, weakly compact set, then there exists $\bar{g}\in L^1(T,Y)$ and a subsequence of $\{\bar{g}_n\}$ such that

$$\bar{g}_{n_k}\xrightarrow{w}\bar{g}$$

Without loss of generality, we can assume $\bar{g}_n\xrightarrow{w}\bar{g}$. By weak semicontinuity of norm, we get

$$\|f-\overline{g}\|\leqslant\liminf\{\|g_n-\overline{g_n}\|\}=\lim\mathrm{dist}(g_n,L^1(T,Y))=\mathrm{dist}(f,L^1(T,Y))$$

i.e. $\overline{g}$ is a best approximant of $f's$ in $L^1(T,Y)$.

We'll prove $\overline{g}$ is a best approximant of $f's$ in $L^p(T,Y)$. Since $\overline{g_n}\xrightarrow{w}\overline{g}$ then there exists a convex combination of $\{\bar{g}_n\}$, such that

$\sum_{i\in I_n}a_i g_i\xrightarrow{L^1}\overline{g}$ where $a\geqslant 0$, $\sum_{i\in I_n}a_i=1$ $I_n=\{i: p_n\leqslant i\leqslant p_{n+1}\}$, and $\{p_n\}$ is an increasing integer number sequence. Set $y_n=\sum_{i\in I_n}a_i\overline{g_i}$ then $\{y_n\}$ is L^1-Cauchy sequence and $\overline{g_n}\in L^p(T,Y)$, so is $\{y_n\}$ in L^p-norm.

In fact:

$$\int_T\|y_{n+p}(t)-y_n(t)\|\mathrm{dm}(t)\to 0$$

By Lebesgue theorem, $\|y_{n+p}(t)-y_n(t)\|\to 0$ in measure sense.

Apparently, $\|y_{n+p}(t)-y_n(t)\|^p\to 0$ (in measure sense) and $\{y_n\}\subset L^p(T,Y)$. So $\left(\int_T\|y_{n+p}(t)-y(t)\|^p\mathrm{d}m(t)\right)^{1/p}\to 0$ i.e. $\{y_n\}$ is L^p-Cauchy sequence. Then there exists $g'_1\in L^p(T,Y)$ such that $\|y_n-g'_1\|_p\to 0$. By Hölder inequality

$$\|y_n-g'_1\|_1\leqslant\|y_n-g'_1\|_p(m(T))^{1/q}\to 0\quad \text{i.e. } y_n\xrightarrow{L^1}g'_1,\ \text{but}\ \ y_n\xrightarrow{L^1}g$$

so $\overline{g}=g'_1$ a.e. $g'_1\in L^p$ then $\bar{g}\in L^p$. We get:

$$\|f-\overline{g}\|_p\leqslant\|f-\sum_{i\in I_n}a_i\overline{g}_i\|_p+\|\sum_{i\in I_n}a_i\bar{g}_i-\bar{g}\|_p\leqslant\sum_{i\in I_n}a_i(\|f-\bar{g}_i\|_p+\|y_n-\bar{g}\|_p$$

$$\leqslant\sum_{i\in I_n}a_i(\|f-g_i\|_p+\|g_i-\bar{g}_i\|_p)+\|y_n-\overline{g}\|_p$$

$$\|f-\bar{g}\|_p < \mathrm{dist}_p(f, L^p(T, Y)) \text{ as } n \to \infty$$

i.e. g is a best approximant of f's in $L^p(T,Y)$. ■

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Banach空间中的几个逼近问题

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本文讨论了Banach空间的一些逼近问题。给出了两个子空间的和可近的充要条件，对已知结果的推广，给出了连续函数空间和$L^p(1\leqslant p<\infty)$空间中的两个逼近定理。