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Smooth Embeddings of 2-Spheres in Manifolds*

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1. Introduction

Let g be the standard generator of $H_2(CP^2;Z)$ and h the standard generator of $H_2(-CP;Z)$. It is well known that $pg+qh\in H_2(CP^2\#(-CP^2);Z)$ can be represented by a smoothly embedded 2-sphere provided $|p|,\ |q|\leqslant 2$ or $||p|-|q||\leqslant 1$, where g and h are the images of the standard generators in $H_2(CP^2\#(-CP^2);Z)$, In this paper we consider the connected sum of CP^2 and several $(-CP^2)'s$.

Let M be $CP^2\#(-CP^2)\#P_1\#P_2\#\cdots\#P_m$, where $P_1,\cdots,P_m$ are m copies of $(-CP^2)$. Let $g,h,\ g_1,\cdots,g_m$ be the images of the standard generators of $H_2(CP^2;Z)$, $H_2(-CP^2;Z)$, $H_2(P_1;Z),\cdots,H_2(P_m;Z)$ in $H_2(M;Z)$ respectively.

Let $\xi=pg+qh+\sum_{i=1}^{m}r_ig_i$ be an element of $H_2(M;Z)$ with $|p|>|q|$, $\sum_{i=1}^{m}r_i^2=p^2-q^2-1$ and $r_i\neq 0,\ i=1,\cdots,m$. We have

Theorem 1. Suppose $p^2-q^2\geqslant 8$, $|p|-|q|\geqslant 2$ and $2(m-1)>p^2-q^2$. Then ξ can not be represented by a smoothly embedded 2-sphere.

Corollary. Suppose $p^2-q^2\geqslant 8$, $|p|-|q|\geqslant 2$ and $m\leqslant p^2-q^2-1$. Then $\eta=pg+qh+\sum_{i=1}^{m}g_i\in H_2(M;Z)$ can not be represented by a smoothly embedded 2-sphere.

Theorem 2. The homology class $pg+qh\in H_2(CP^2\#(-CP^2);Z)$ can be represented by a smoothly embedded 2-sphere if and only if $|p|,\ |q|\leqslant 2$ or $||p|-|q||\leqslant 1$.

2. Proof of Theorem 1

Proof of Theorem 1: For convenience we assume: $q\geqslant 0$, $p,r_i>0,\ i=1,\cdots,m$. The other cases is similar.

If ξ can be represented by a smoothly embedded 2-sphere S, let A be a tubular neighbourhood of S in M. A is the total space of a normal 2-disc-bundle $\pi: A\to S$ in M. The restriction to the boundary $\pi|_{\partial A}: \partial A\to S$ is the associated

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S' bundle. Since the Euler number of this S^1-bundle is $S\cdot S=\xi\cdot\xi=p^2-q^2-\sum_{i=1}^{m}\cdot r_i^2=1$, it is the 1-Hopf fibration. Thus $\partial A=S^3$. Let $N=(M\text{-Int }A)\bigcup_f D^4$, where D^4 is a 4-ball and f a diffeomorphism from ∂D onto $\partial A=\partial(M\text{-Int }A)$. N is a closed smooth 4-manifold and is simply connected. We get

$$M=N\#CP^2.$$

Let S_x be the intersection form of manifold X. Thus we have

$$S_M\sim S_N\oplus S_{CP^2}.$$

But $S_M=\langle 1\rangle\oplus(m+1)\langle -1\rangle$ and $S_{CP^2}=\langle 1\rangle$. So S_N is negatively definite. By Danaldson theorem ([1]), we obtain

$$S_N\sim(m+1)\langle -1\rangle.$$

Therefore there exist $2(m+1)$ homology classes in $H_2(N;Z)$ with selfintersection number -1. The images in $H_2(M;Z)$ of them have selfintersection number -1 as well and have intersection number O with ξ.

Let $a=xg+yh+\sum_{i=1}^{m}z_ig_i\in H_2(M;Z)$ such that $a\cdot\xi=0$ and $a\cdot a=-1$. Then x, $y,z_1,\cdots,z_m$ satisfying following Diophantine equations

$$\begin{cases} px-qy-\sum_{i=1}^{m}r_iz_i=0 \\ x^2-y^2-\sum_{i=1}^{m}z_i^2+1=0 \end{cases} \qquad (1)$$

We will show that integral solutions of (1) are less than $2(m+1)$.

Discarding z_i's which are zero and renumbring the nonzero ones and corresponding $r_j's$, $z_1,\cdots,z_s$; $r_1,\cdots,r_s$, $0\leqslant s\leqslant m$, we may eliminate x to obtain

$$(p^2-q^2)y^2-2q(\sum_{i=1}^{s}r_iz_i)y-(\sum_{i=1}^{s}r_iz_i)^2+p^2(\sum_{i=1}^{s}z_i^2-1)=0 \qquad (2)$$

As a quadratic equation of y its discriminant is

$$\Delta=p^2(\sum_{i=1}^{s}r_iz_i)^2-p^2(p^2-q^2)(\sum_{i=1}^{s}z_i^2-1)$$

Set $\delta=\Delta/p^2$, we have:

i). $s=0$. (2) becomes $(p^2-q^2)y^2-p^2=0$ and has at most two integral solutions.

ii). $s=1$. For $z_1=\pm1$, (2) becomes $(p^2-q^2)y^2\mp 2qr_1y-r_1^2=0$. Its solutions are $\pm\frac{r_1}{p+q}$ and $\pm\frac{r_1}{p-q}$. Since $p+q\geqslant p$ and $r_1^2<p^2-q^2$, $\frac{r_1}{p+q}$ is not integral number. $\frac{r_1}{p-q}$ is an integral number if and only if $(p-q)|r_1$. Since $p-q\geqslant 2$ and there are at lest two $r_i's$ which equal 1 (otherwise, $\sum_{i=1}^{m}r_i^2\geqslant\sum_{i=1}^{m-1}r_i^2+1\geqslant 4(m-$

$1)+1>p^2-q^2$, contradiction), (2) has at most $2(m-2)$ integral solutions.

For $z_1^2\geqslant 4$, since $2(m-1)>p^2-q^2$, we have $r_1<\frac{3}{4}(p^2-q^2)<\frac{z_1^2-1}{z_1^2}(p^2-q^2)$ and hence $\delta=r_1^2z_1^2-(p^2-q^2)(z_1^2-1)<0$. (2) has no solutions.

iii). $2\leqslant s\leqslant m-1$. Suppose $p^2-q^2-\sum_{i=1}^{s}r_i^2=k$. We have $k\geqslant m-s+1\geqslant 2$ and $\sum_{i=1}^{s}z_i^2\geqslant s\geqslant 2$. Thus $k\cdot\sum_{i=1}^{s}z_i^2\geqslant(m-s+1)\cdot s\geqslant 2(m-1)>p^2-q^2$. It follows

$$\frac{\sum_{i=1}^{s}r_i^2}{p^2-q^2}=\frac{p^2-q^2-k}{p^2-q^2}<\frac{k\cdot\sum_{i=1}^{s}z_i^2-k}{k\cdot\sum_{i=1}^{s}z_i^2}=\frac{k(\sum_{i=1}^{s}z_i^2-1)}{k\sum_{i=1}^{s}z_i^2}=\frac{\sum_{i=1}^{s}z_i^2-1}{\sum_{i=1}^{s}z_i^2}.$$

and hence $\delta=(\sum_{i=1}^{s}r_iz_i)^2-(p^2-q^2)(\sum_{i=1}^{s}z_i^2-1)\leqslant(\sum_{i=1}^{s}r_i^2)(\sum_{i=1}^{s}z_i^2)-(p^2-q^2)(\sum_{i=1}^{s}z_i^2-1)$

$$=(p^2-q^2)(\sum_{i=1}^{s}z_i^2)\left(\frac{\sum_{i=1}^{s}r_i^2}{p^2-q^2}-\frac{\sum_{i=1}^{s}z_i^2-1}{\sum_{i=1}^{s}z}\right)<0.$$

(2) has no solutions.

iv). $s=m$. Suppose $\xi=pg+qh+\sum_{i=1}^{n}r_ig_i+\sum_{i=n+1}^{m}g_i$, i.e. $r_i=1$ when $i=n+1,\cdots,m$. From ii), we have $n\leqslant m-2$.

If $\{z_i\}_{i=1}^{m}=\{r_i\}_{i=1}^{m}$ or $\{z_i\}_{i=1}^{m}=\{-r_i\}_{i=1}^{m}$, then $\delta=1$.

We will show that if $\{z_i\}_{i=1}^{m}\neq\pm\{r_i\}_{i=1}^{m}$, then $\delta<0$, hence (2) has at most two solutions.

It is easy to see that if $\mathrm{sign}(z_i)\neq\mathrm{sign}(z_j)$ for some i,j, then $\delta<0$. So we assume $\mathrm{sign}(z_i)=\mathrm{sign}(z_j)$, $1\leqslant i,j\leqslant m$. Without loss of generality, we assume $z_i>0$, $i=1,\cdots,m$.

a) $z_i\geqslant 2$, for some i, $n+1\leqslant i\leqslant m$, say $z_m\geqslant 2$. By iii), $(\sum_{i=1}^{m-1}r_iz_i)^2-(p^2-q^2)\cdot$ $\cdot(\sum_{i=1}^{m-1}z_i^2-1)<0$.

If $\sum_{i=1}^{m-1}r_iz_i-(p^2-q^2)\geqslant -2$, then $\sum_{i=1}^{m-1}r_i^2\sum_{i=1}^{m-1}z_i^2\geqslant(\sum_{i=1}^{m-1}r_iz_i)^2\geqslant(p^2-q^2-2)^2$. Since $\sum_{i=1}^{m-1}r_i^2=p^2-q^2-2$ we have $\sum_{i=1}^{m-1}z_i^2\geqslant p^2-q^2-2$ and $\sum_{i=1}^{m}z_i^2=\sum_{i=1}^{m-1}z_i^2+z_m^2\geqslant p^2-q^2-2+z_m^2$ $\geqslant p^2-q^2-2+4>p^2-q^2$. It follows

$$\frac{\sum_{i=1}^{m}r_i^2}{p^2-q^2}=\frac{p^2-q^2+1}{p^2-q^2}<\frac{\sum_{i=1}^{m}z_i^2-1}{\sum_{i=1}^{m}z_i^2}.$$

Thus $\delta=(\sum_{i=1}^{m} r_i z_i)^2-(p^2-q^2)(\sum_{i=1}^{m} z_i^2-1)\leqslant(\sum_{i=1}^{m} r_i^2)(\sum_{i=1}^{m} z_i^2)-(p^2-q^2)(\sum_{i=1}^{m} z_i^2-1)<0.$

If $(\sum_{i=1}^{m-1} r_i z_i)-(p^2-q^2)\leqslant-3$, then $\delta=(\sum_{i=1}^{m} r_i z_i)^2-(p^2-q^2)(\sum_{i=1}^{m} z_i^2-1)=(\sum_{i=1}^{m-1} r_i z_i + z_m)^2-(p^2-q^2)(\sum_{i=1}^{m-1} z_i^2+z_m^2-1)=(\sum_{i=1}^{m-1} r_i z_i)^2+2z_m(\sum_{i=1}^{m-1} r_i z_i)+z_m^2-(p^2-q^2)(\sum_{i=1}^{m-1} z_i^2-1)-z_m^2(p^2-q^2)<2z_m(p^2-q^2-3)-z_m^2(p^2-q^2-1)\leqslant z_m^2(p^2-q^2-3)-z_m^2(p^2-q^2-1)<0.$

b) $z_{n+1}=\cdots=z_m=1$. Let $s_i=r_i-z_i$, $i=1,\cdots,n$. Since $\{z_i\}\neq\{r_i\}$, we have $s_i\neq 0$ for some i.

$\delta=(\sum_{i=1}^{m} r_i z_i)^2-(p^2-q^2)(\sum_{i=1}^{m} z_i^2-1)=(\sum_{i=1}^{m} r_i^2-\sum_{i=1}^{n} r_i s_i)^2-(p^2-q^2)(\sum_{i=1}^{m} r_i^2-1-2\sum_{i=1}^{n} r_i s_i+\sum_{i=1}^{n} s_i^2)=(\sum_{i=1}^{m} r_i^2)^2-(p^2-q^2)(\sum_{i=1}^{m} r_i^2-1)-2(\sum_{i=1}^{m} r_i^2)(\sum_{i=1}^{n} r_i s_i)+(\sum_{i=1}^{n} r_i s_i)^2++2(p^2-q^2)(\sum_{i=1}^{n} r_i s_i)-(p^2-q^2)\sum_{i=1}^{n} s_i^2=1+2\sum_{i=1}^{n} r_i s_i+(\sum_{i=1}^{n} r_i s_i)^2-(p^2-q^2)\sum_{i=1}^{n} s_i^2=(1++\sum_{i=1}^{n} r_i s_i)^2-(p^2-q^2)(\sum_{i=1}^{n} s_i^2+1-1)<0.$

The last inequality is from iii).

By i), ii), iii) and iv), we have that the solutions of (2) hence (1) are at most $2m$. This proves Theorem 1.

The following examples show that the restriction for m and $r_i' s$ in Theorem 1 is needed.

Example 1. The notations is as above. Taking $m=5$, $p=4$, $q=2$, $r_1=r_2=2$, $r_3=r_4=r_5=1$ Then $\xi\in H_2(M;Z)$ can be represented by a smoothly embedded 2-sphere in M. This can be proved by using a geometric construction as the proof of Theorem 2 in the next section.

Example 2. Taking $p=5, q=2$, $r_1=\cdots=r_5=2$. Then $\xi\in H_2(M;Z)$ can be represented by a smoothly embedded 2-sphere.

Proof of corollary: If $\eta=pg+qh+\sum_{i=1}^{m} g_i\in H(M;Z)$ can be represented by a smoothly embedded 2-sphere. Let $M'=M\#P_{m+1}\#\cdots\#P_{p^2-q^2-1}$, where $P_{m+1},\cdots,P_{p^2-q^2-1}$, are copies of $(-CP^2)$, and $\xi=pg+qh+\sum_{i=1}^{m} g_i+\sum_{i=m+1}^{p^2-q^2-1} g_i$, then ξ can be represented by a smoothly embedded 2-sphere as well. But $2(p^2-q^2-1)>p^2-q^2$ This is a contradiction to Theorem 1.

Remark. It is easy to see that Theorem 1 and the corollary is also true if we change CP^2 and $(-CP^2)$.

3. Proof of Theorem 2

The "only if" part of Theorem 2 is a special case of Corollary (changing CP and $(-CP)$ if $|q|<|p|$). The "if" part is well known provided $|p|$, $|q|\leqslant 2$, and the other cases can be found in [3](the proof of Proposition 6.6). Here we give an intuitive proof.

Without loss of generality we assume $p>0$.

Taking a copy of $CP^2=\{[z_1,z_2,z_3]\}$ with preferred orientation. CP^2 can be considered as the union $U\cup V$, where $V=\{[z_1,z_2,z_3]\in CP^2;\ z_3=1,\ |z_1|^2+|z_2|^2\leqslant 1\}$ is a 4-disc and $U=CP-\text{Int}V$ is a tubular neighbourhood of the complex projective line $CP^1=\{[z_1,z_2,z_3]\in CP^2;z_3=0\}=S^2\subset CP^2$. Let $A_\theta=\{[z'\cos\theta, z'\sin\theta, z_3]\in CP^2\}$. Clearly $A_\theta\cong CP^1=S^2$ for θ fixed. Take p distinct $\theta_1,\cdots,\theta_p$ in $[0,\pi)$. $A_{\theta_1},\cdots,A_{\theta_p}$ intersect at point $[0,0,z_3]\in V$. The projection $f:U\to CP^1$ is defined by $f([z_1,z_2,z_3])=[z_1,z_2,0]$. We regard that $D_i=U\cap A_{\theta_i}, i=1,\cdots,p$. The discs are oriented so that their intersection numbers with CP^1 are all equal to 1. The oriented dises cut $\partial U=S^3$ transversely at an oriented link L consisting of oriented knots $\partial D_1,\cdots,\partial D_p$.

Take another copy of CP^2 and copies of all the things above. We use a $'$ to denote the new things. But this time, the orientations of CP^2, discs and link are the opposite ones.

We can glue U and U' by identifying $\partial U=S^3$ and $\partial U'=S^3$ identically and obtain the connected sum $CP^2\#(-CP^2)$ and smoothly embedded 2-spheres (after smoothing the corners) $D_1\cup D_1',\cdots,D_p\cup D_p'$, which are disjoint mutually. Then we can easily pipe them to get a smoothly embedded 2-sphere which represents the homology class $pg-qh$.

Take the third copy of CP^2 and the all things above. Only choose the opposite orientation on CP^2 and the same on discs and link. We use a $''$ to denote them. Let $\text{id}: U\to U''$ be the copy identifying mapping.

Let $t:\partial U—\partial U$ be defined by $t([z_1,z_2,1])=[i\overline{z_1}, i\overline{z_2},1]$. Clearly t is in SO (4) and is isotopic to the identity. Notice that t sends ∂D_i to itself and reverses the orientation of it for $i=1,\cdots,p$ simultaneously, i.e. t sends the oriented link L to itself orientation-reversely.

Then gluing U and U'' by the composition $\text{id}\circ t$ along ∂U and $\partial U''$, we obtain the connected sum $CP^2\#(-CP^2)$ and disjoint smoothly embedded 2-sphere $D_1\cup D_1'',\cdots,D_p\cup D_p''$, too. And we pipe them to get a smoothly embedded 2-sphere, which represents the homology class $pg+qh$.

Moreover, notice that CP^1 in U'/or U'' intersects each D'/or D'' at just one point. By piping CP^1/or $-CP^1$ with $D_1\cup D_1',\cdots,D_p\cup D_p'$/or $D_1\cup D_1'',\cdots,D_p\cup D_p''$

in $CP^2\#(-CP^2)$ at intersection point, we thus obtain a smoothly embedded 2 sphere which represents the homology class $pg\pm(p\pm1)h$ in $H_2(CP^2\#(-CP^2);Z)$.

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