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On the Rate of Convergence of the Feller-Trotter Probability Type Operator for Functions of Bounded Variation*

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I. Introduction

Let (Ω, Ξ, P) be a probability space and $\{X_n\}$ be a sequence of independent random variables with identified distribution as X's. The expectation $E(X)$ of X is $x(x \in J)$, where J is an infinite or a finite subinterval on $(-\infty, +\infty)$. $F_{S_n}(t)$ is the distribution function of S_n. The Feller-Trotter probability type operator is

$$T_n(f, x) = \int_{-\infty}^{+\infty} f(\frac{t}{n})\, dF_{S_n}(t) \qquad x \in J. \tag{1.1}$$

This operator was introduced by *H. F.* Trotter in 1959 to establish the Central Limit Theorem (see [1]), but its approximation properties were not investigated until 70's. Up to now, T. Hermann, P. L. Butzer, L. Haln, R. A. Khan, G. A. Anastassiou, etc. have studied its approximation properties for some continuous functions. In this paper, we study its approximation properties for functions of so-called bounded variation. Our main result is included in Theorem 1 and the Remark of Theorem 1 tells us theorem 1 is essentially best possible. As many common operators can be deduced from the Feller-Trotter probability type operator, properties of many common operators can be easily derived from Theorem 1. Therefore, in Section 4, we obtain the rate of convergence of Baskakov operator, Gamma operator, Post-Gamma operator, Szasz-Mirakjan operator and Gauss Weierstrass operator for functions of bounded variation. Theorem 3 (the result of Szasz-Mirakjan opetapor) generalizes and improves F. Cheng's result (see [3]) and other results are new.

Throughout this paper, BV(J) denotes the set of functions with bounded variation on every finite subinterval on J and the distribution of X is not degenerate and X's covariance is DX.

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2. Lemmas

Given that $M(t)$(the moment generating function of X, i.e. $M(t)=E(e^{tX})$ is defined in a neighbourhood of 0 and $m(a)=\inf_t(m(t)e^{-at})$. Therefore, if $a\neq E(X)$, we have $m(a)<1$ (see [4]) & the following lemmas

Lemma 1

i) if $a\leqslant E(X)$, then $P(S_n\leqslant na)\leqslant[m(a)]^n$

ii) if $a\geqslant E(X)$, then $P(S_n\geqslant na)\leqslant[m(a)]^n$

Lemma 2 Let a be a non-negative constant. If n is sufficiently large, we have

i) if $a>E(X)$, then $\int_a^{\infty}e^{a|t|}dF_{S_n}(nt)<3e^{2a|a|}[m(a)]^n$;

ii) if $a<E(X)$, then $\int_{-\infty}^{a}e^{a|t|}dF_{S_n}(nt)<3e^{2a|a|}[m(a)]^n$.

Lemma 3 Let $E(X)=x$ and $b<x<d$, where b, d are real numbers. Then

i) $K_n(t)\leqslant DX/n(t-x)^2$ for $b\leqslant t<x$, where

$$K_n(t)=\begin{cases}F_{S_n}(nt)-F_{S_n}(nb-), & t\in(b,x]\\ 0 & t=b\end{cases}$$

ii) $Q_n(t)\leqslant DX/(n(t-x)^2)$ for $x<t\leqslant d$, where

$$Q_n(t)=\begin{cases}1-F_{S_n}(nt-), & t\in[b,d)\\ 0 & t=d\end{cases}$$

Remark The proof of Lemma 1 see [4, Theorem 1], Lemma 2 may be proved from Lemma 1 with methods similar to the proof in [5, P. 330] and Lemma 3 may be proved from properties of covariance DX.

3. General Results and Proofs

Theorem 1 Assume that the moment generating function $M(t)$ exists in a neighbourhood of O and $E(X)=x$ $(x\in J, x\neq 0)$. If $f\in BV(-\infty,+\infty)$ and $f(t)=O(e^{A|t|})$ $(|t|\to\infty, A>0)$, for sufficiently large n we have

$$|T_n(f,x)-(f(x+)+f(x-))/2|\leqslant\frac{1}{n}\left(\frac{3DX}{x^2}+1\right)\sum_{k=1}^{n}\bigvee_{x-|x|/\sqrt{k}}^{x+|x|/\sqrt{k}}(g_x)+O(1)e^{4A|x|}v^n$$
$$+\sqrt{2/(n\pi)}\,hQ(x)\,|f(x+)-f(x-)| \qquad (3.1)$$

Where $v=\mathrm{Max}\{m(0), m(2|x|)\}<1$, $Q(x)=|E(X-x)^3|/(DX)^{3/2}+1/DX$. If X is a random variable with lattice point distribution and its largest distribution index is larger than 1, h is the largest distribution index, otherwise, $h=1$. g is defined as

$$g_x(t)=\begin{cases}f(t)-f(x+) & t>x\\ 0 & t=x\\ f(t)-f(x-) & t<x\end{cases} \qquad (3.2)$$

Remark The rate of (3.1) cannot be improved.

Proof of Theorem 1 i) At first let's suppose that $T_n(f, x)$ exists. For

$$f(t)-(f(x+)+f(x-))/2=g_x(t)+(f(x+)-f(x-))/2\operatorname{sgn}(t-x),$$

we have

$$T_n(f,x)-(f(x+)+f(x-))/2=T_n(g_x,x)+T_n(\operatorname{sgn}(t-x),x)(f(x+)-f(x-))/2 \tag{3.3}$$

Therefore, using [6, Th.2in § 42 and Th.1 in § 43] we obtain

$$|F_{S_n}(nx)-1/2|<\sqrt{2/(n\pi)}\,hQ(x), \qquad |F_{S_n}(nx-)-1/2|<\sqrt{2/(n\pi)}\,hQ(x).$$

By (1.1), (3.3) and the inequalities above, we have

$$|T_n(f,x)-(f(x+)+f(x-))/2|<\sqrt{2/(n\pi)}\,hQ(x)|f(x+)-f(x-)|+|T_n(g_x,x)| \tag{3.4}$$

Using replacement of variables and (1.1), we have

$$T_n(g_x,x)=\int_{-\infty}^{+\infty}g_x\left(\frac{t}{n}\right)\mathrm{d}F_{S_n}(t)=\int_{-\infty}^{+\infty}g_x(t)\,\mathrm{d}F_{S_n}(nt)\overset{\text{def}}{=\!=}L_{1n}+L_{2n}+M_n+R_{1n}+R_{2n} \tag{3.5}$$

where $L_{1n}, L_{2n}, M_n, R_{1n}, R_{2n}$ denote the Lebesgue-Stieltjes integrals on intervals $(-\infty, x-|x|)$, $[x-|x|, x-|x|/\sqrt{n}]$, $(x-|x|/\sqrt{n}, x+|x|/\sqrt{n})$, $[x+|x|/\sqrt{n}, x+|x|]$ and $(x+|x|, \infty)$ in turn. By Lemma 1 and Lemma 3, using the methods in [3], when n is sufficiently large, we obtain

$$|L_{2n}|<2DX/(nx^2)\sum_{k=1}^{n}\bigvee_{x-|x|/\sqrt{k}}^{x}(g_x),$$

$$|M_n|<1/n\sum_{k=1}^{n}\bigvee_{x-|x|/\sqrt{k}}^{x+|x|/\sqrt{k}}(g_x)$$

$$|R_{2n}|<3DX/(nx^2)\sum_{k=1}^{n}\bigvee_{x}^{x+|x|/\sqrt{k}}(g_x).$$

From (3.4) - (3.5) and the inequalities above, we have

$$|(T_n(f,x)-(f(x+)+f(x-))/2|<1/n(3DX/x^2+1)\sum_{k=1}^{n}\bigvee_{x-|x|/\sqrt{k}}^{x+|x|/\sqrt{k}}(g_x)+|L_{1n}|$$
$$+|R_{1n}|+\sqrt{2/(n\pi)}\,hQ(x)|f(x+)-f(x-)| \tag{3.6}$$

We should notice that it's deduced from the condition that $T_n(f, x)$ exists but not from the condition that $f(t)=o(e^{A|t|})$. This is useful in Theorem 3.

Because $M(t)$ exists on a neighbourhood of o, the moment generating function of S_n exists on the neighbourhood. Furthermore, $f(t)=o(e^{A|t|})$. Thus, $T_n(f,x)$ is defined for sufficiently large n. Consequently, (3.6) is true.

ii) From (3.2) and $f(t)=o(e^{A|t|})$, we have $|g_x(t)|<Ke^{A|t|}$, where K is a positive number. Thus, by Lemma 2, we get

$$|R_{1n}|<\int_{x+|x|}^{\infty}Ke^{A|t|}\mathrm{d}F_{S_n}(nt)<3Ke^{2A(x+|x|)}[m(x+|x|)]^n$$

$$|L_{1n}| < \int_{-\infty}^{x-|x|} K e^{A|t|} \mathrm{d} F_{S_n}(nt) < 3K e^{2A(|x|-x)} [m(x-|x|)]^n$$

Consequently, $$|L_{1n}| + |R_{1n}| = O(1) e^{4A|x|} v^n. \qquad (3.7)$$

By (3.6) - (3.7), Theorem 1 is proved.

Proof of the Remark on Theorem 1.

Let $f(t) = |t-x|$ $(x>0)$. By (3.3) and (3.1) we have $g_x = f$ and

$$|T_n(f, x) - f(x)| < 2(4DX/x^2+1)|x|/n \sum_{k=1}^{n} 1/\sqrt{k} < 4|x|(1+4DX/x^2)/\sqrt{n} \qquad (3.8)$$

But when X's distribution function is $P(x)$(Poisson distribution), T_n is Szasz-Mirakjan operator S_n (see [7, P. 313]) and we have (see [3])

$$|S_n(f, x) - f(x)| \geq \frac{1}{\sqrt{n}} [2x/(\pi e^4)]^{\frac{1}{2}} \qquad (3.9)$$

By (3.8) - (3.9), we know that the rate of (3.1) cannot be improved.

4. Application of the General Theorem

a. If X's distribution is geometric distribution $G(1/(1+x))$, Gamma distribution $\Gamma(1/x, 1)$ and $\Gamma(1, x)$ $(x>0)$ (see [8, P. 62]), the Feller-Trotter probability type operator is Baskakov operator V_n, Post-Gamma operator H_n and Gamma operator G_n in turn (see [7, P.313—315]):

$$V_n(f, x) = (1+x)^{-n} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \binom{n+k-1}{k} (x/(1+x))^k, \quad x>0 \qquad (4.1)$$

$$H_n(f, x) = \frac{1}{(n-1)!\, x^n} \int_0^{\infty} f\left(\frac{t}{n}\right) t^{n-x} e^{-t/x} \mathrm{d}t \qquad x>0 \qquad (4.2)$$

$$G_n(f, x) = \frac{n^{nx}}{\Gamma(nx)} \int_0^{\infty} f(t) t^{nx-1} e^{-nt} \mathrm{d}t \qquad x>0 \qquad (4.3)$$

By calculating $E(X)$, DX, $Q(x)$, h and v of the three distributions and replacing these in Theorem 1 with the results, we obtain

Theorem 2 If $f \in BV[0, +\infty)$ and $f(t) = O(e^{at})$ $(t \to \infty, a>0)$, for every positive x, when n is sufficiently large, we have

$$|H_n(f, x) - (f(x+)+f(x-))/2| < 4/n \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(1) e^{4ax} (2/e)^n + \sqrt{2/(n\pi)}\,(2+1/x)\,|f(x+)-f(x-)| \qquad (4.4)$$

$$|G_n(f, x) - (f(x+)+f(x-))/2| < (3+x)/(nx) \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(1) e^{4nx} (2/e)^n + \sqrt{18/(n\pi x)}\,|f(x+)-f(x-)| \qquad (4.5)$$

$$|V_n(f, x) - (f(x+)+f(x-))/2| < (3+4x)/(nx) \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(1) e^{4ax} v^n + \sqrt{8(1+x)/(nx\pi)}\,|f(x+)-f(x-)| \qquad (4.6)$$

where g_x is defined by (3.2) and $v = 2((2x+1)/(2x+2))^{2x+1} < 1$ in (4.6)

Remark The Gamma and the Post-Gamma operator G_n and H_n defined in (4.2) and (4.3) are all different from the Gamma operator in [10].

b. If X's distribution function is $P(x)$ (Poisson distribution), T_n is Szasz-Mirakjan operator S_n(see [7, P.313]):

$$S_n(f,x)=e^{-nx}\sum_{k=0}^{\infty}f(\frac{k}{n})(nk)^k/k! \qquad x\in(0,\infty) \tag{4.7}$$

Through calculation we know that $E(X)=DX=x$, $Q(x)=2/\sqrt{x}$, $h=1$ and $M(t)$ exists for $t\in R$. Furthermore, we know that $S_n(t^{at},x)$ exists for $a>0$. Therefore, if $f(t)=O(t^{at})(t\to\infty)$ and $f\in BV[0,\infty)$, $S_n(f,x)$ exists and $g_x(t)=O(t^{at})(t\to\infty)$ (see (3.2)). By [3, lemma 6], we have

$$|R_{1n}|=\left|\int_{t>2x}g_x(t)\,\mathrm{d}F_{S_n}(nt)\right|=\left|\int_{t>2nx}g_x(\frac{t}{n})\,\mathrm{d}F_{S_n}(t)\right|$$

$$< M\sum_{k>2nx}(\frac{k}{n})^{ak/n}e^{-nx}(nx)^k/k! < 3M/\sqrt{4nx\pi}\,(4x)^{4ax}(e/4)^{nx} \tag{4.9}$$

By (3.6), (4.9) and $L_{1n}=0$, we obtain

Throrem 3 If $f\in BV[0,\infty)$ and $f(t)=O(t^{at})(t\to\infty,\ a>0)$, for every positive x, when n is sufficiently large, we have

$$\begin{aligned}|S_n(f,x)-(f(x+)+f(x-))/2| &< (3+x)/(nx)\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)\\ &+\sqrt{8/nx\pi}\,|f(x+)-f(x-)|\\ &+O(1)1/\sqrt{nx}(4x)^{4ax}(e/4)^{nx}\end{aligned} \tag{4.10}$$

where g_x is defined in (3.2).

Remark Theorem 3 generalizes and improves the results in [3].

c. If X's distribution function is normal distribution $N(x,\sqrt{1/2})$, T_n is the Gauss-Weierstrass operator W_n (see [2]):

$$W_n(f,t)=\sqrt{n/\pi}\int_{-\infty}^{+\infty}f(t)e^{-n(t-x)^2}\mathrm{d}t, \qquad x\in(-\infty,+\infty) \tag{4.11}$$

Making out $E(X)$, DX, $Q(x)$, h and v in Theorem 1, we obtain

Theorem 4 If $f\in BV(-\infty,+\infty)$ and $f(t)=O(e^{a|t|})(t\to\infty,\ a>0)$, for every positive x, when n is sufficiently large, we have

$$\begin{aligned}|W_n(f,x)-(f(x+)+f(x-))/2| &< (3+2x^2)/(2nx^2)\sum_{k=1}^{n}\bigvee_{x-|x|/\sqrt{k}}^{x+|x|/\sqrt{k}}(g_x)\\ &+2/\sqrt{nx}\,|f(x+)-f(x-)|+O(1)e^{4ax}e^{-nx^2}\end{aligned} \tag{4.12}$$

where g_x is defined by (3.2).

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Feller-Trotter概率型算子对有界变差函数的收敛速度

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摘 要

本文运用概率方法研究了Feller-Trotter概率型算子对有界变差函数的收敛速度．由于该算子包括许多常见的算子，从而由关于该算子的一般结论可导出许多常见算子对有界变差函数的收敛速度．作为一般结论的应用，本文列举了Baskakov算子、Szasz-Mirakjan算子、Gauss-Weierstrass算子、Gramma 算子、Post-Gamma 算子对有界变差函数的收敛速度．其中，关于Szasz-Mirakjan算子的结论推广并改进了Fuhua Cheng 的结论，其它结论是作者首次得到．