Journal of Mathematical Research and Exposition
Vol. 10. No.2, May, 1990

On p-Valent Functions With Negative And Missing Coefficients*

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Abstract

Let $P_k(A, B, p, a)$ be the class of functions $f(z)=z^p-\sum_{n=k+1}^{\infty}|a_n|z^n$, $k>p>1$, analytic and p-valent in the unit disc $U=\{z:|z|<1\}$ and satisfying

$$\left|\frac{\frac{f'(z)}{z^{p-1}}-p}{[pB+(A-B)(p-a)]-B\frac{f'(z)}{z^{p-1}}}\right|<1, \qquad z\in U,$$

where $-1<B<A<1$ and $0<a<p$.

In this paper we obtain coefficient estimate, distortion, closure theorems and radius of convexity for the class $P_k(A, B, p, a)$ under the assumption $-1\leqslant B<0$. We also obtain class preserving integral operators of the form

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt, \qquad c>-p$$

for the class $P_k(A, B, p, a)$. Conversely when $F(z)\in P_k(A, B, p, a)$, radius of p-valence of $f(z)$ has been determined.

I. Introduction.

Let S_p denote the class of functions of the form $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ which are analytic and p-valent in the unit disc $U=\{z:|z|<1\}$. For $-1\leqslant B<A\leqslant 1$ and $0\leqslant a<p$, let $P^*(A, B, p, a)$ be the class of those functions $f(z)$ of S_p for which $\frac{f'(z)}{z^{p-1}}$ is subordinate to $\frac{p+[pB+(A-B)(p-a)]z}{1+Bz}$. In other words $f(z)\in P^*(A, B, p, a)$, if and only if there exists a function $w(z)$ satisfying $w(0)=0$ and $|w(z)|<1$ for $z\in U$, such that

$$\frac{f'(z)}{z^{p-1}}=\frac{p+[pB+(A-B)(p-a)]w(z)}{1+Bw(z)}, \qquad z\in U. \tag{1.1}$$

The condition (1.1) is equivalent to

* Received Feb. 19, 1989

$$\left|\frac{\dfrac{f'(z)}{z^{p-1}}-p}{[pB+(A-B)(p-a)]-B\dfrac{f'(z)}{z^{p-1}}}\right|<1, \quad z\in U.$$

It is clear from (1.1) that $\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>a$, $0\leqslant a<p$, $z\in U$, and hence the members of $P^*(A, B, p, a)$ are p-valent in U (see Umezawa [5]). Let T_p denote the subclass of S_p consisting of functions analytic and p-valent which can be expressed in the form

$$f(z)=z^p-\sum_{n=k+1}^{\infty}|a_n|z^n \quad (k\geqslant p\geqslant 1).$$

Let us define

$$P_k(A, B, x, a)=P^*(A, B, p, a)\wedge T_p.$$

In this paper, under the assumption $-1\leqslant B\leqslant 0$, we obtain coefficient estimate, distortion and covering theorems and radius of convexity for the class $P_k(A, B, p, a)$. We also obtain the class preserving integral operators of the form

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt, \quad c>-p \tag{1.2}$$

for $P_k(A, B, p, a)$. Conversely, when $F(z)\in P_k(A, B, p, a)$, we determine the radius of p-valence of $f(z)$ defined by (1.2). Lastly, we show that the class $P_k(A, B, p, a)$ is closed under "arithmetic mean" and "convex linear combinations".

Note Throughout this paper we assume that $-1\leqslant B\leqslant 0$ and $k\geqslant p\geqslant 1$.

2. Coefficient estimate

Theorem 1 A function $f(z)=z^p-\sum_{n=k+1}^{\infty}|a_n|z^n$ is in $P_k(A, B, p, a)$ if and only if

$$\sum_{n=k+1}^{\infty}n(1-B)|a_n|\leqslant(A-B)(p-a).$$

This result is sharp.

Proof Let $|z|=1$. Then

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|-\left|[pB+(A-B)(p-a)]-B\frac{f'(z)}{z^{p-1}}\right|$$

$$=\left|-\sum_{n=k+1}^{\infty}n|a_n|z^{n-p}\right|-\left|(A-B)(p-a)+B\sum_{n=k+1}^{\infty}n|a_n|z^{n-p}\right|$$

$$\leqslant\sum_{n=k+1}^{\infty}n(1-B)|a_n|-(A-B)(p-a) \quad (\text{since } -1\leqslant B\leqslant 0)\leqslant 0$$

by assumption.

Hence, by maximum modulus principle, $f(z)\in P_k(A, B, p, a)$.

To prove the converse, let

$$\left|\frac{\frac{f'(z)}{z^{p-1}}-p}{[pB+(A-B)(p-a)]-B\frac{f'(z)}{z^{p-1}}}\right|=\left|\frac{-\sum_{n=k+1}^{\infty}n|a_n|z^{n-p}}{(A-B)(p-a)+B\sum_{n=k+1}^{\infty}n|a_n|z^{n-p}}\right|<1,\ z\in U.$$

Since $|\mathrm{Re}(z)|\leqslant|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=k+1}^{\infty}n|a_n|z^{n-p}}{(A-B)(p-a)+B\sum_{n=k+1}^{\infty}n|a_n|z^{n-p}}\right\}<1. \tag{2.1}$$

Choose values of z on the real axis so that $\frac{f'(z)}{z^{p-1}}$ is real. Upon clearing the denominator in (2.1) and letting $z\longrightarrow 1^-$ through real values, we obtain

$$\sum_{n=k+1}^{\infty}n|a_n|\leqslant(A-B)(p-a)+B\sum_{n=k+1}^{\infty}n|a_n|.$$

This completes the proof of the theorem.

Sharpness follows if we take

$$f(z)=z^p-\frac{(A-B)(p-a)}{n(1-B)}z^n\ ,\quad(n\geqslant k+1,\ k\geqslant p\geqslant 1).$$

3. Distortion properties.

Theorem 2 If $f(z)\in P_k(A,B,p,a)$, then for $|z|=r$

$$r^p-\frac{(A-B)(p-a)}{(k+1)(1-B)}r^{k+1}\leqslant|f(z)|\leqslant r^p+\frac{(A-B)(p-a)}{(k+1)(1-B)}r^{k+1} \tag{3.1}$$

and

$$pr^{p-1}-\frac{(A-B)(p-a)}{(1-B)}r^k\leqslant|f'(z)|\leqslant pr^{p-1}+\frac{(A-B)(p-a)}{(1-B)}r^k. \tag{3.2}$$

All the inequalities are sharp.

Proof Let $f(z)=z^p-\sum_{n=k+1}^{\infty}|a_n|z^n$. Then, it follows from Theorem 1 that

$$\sum_{n=k+1}^{\infty}|a_n|\leqslant\frac{(A-B)(p-a)}{(k+1)(1-B)}.$$

Hence

$$|f(z)|\leqslant r^p+\sum_{n=k+1}^{\infty}|a_n|r^n\leqslant r^p+r^{k+1}\sum_{n=k+1}^{\infty}|a_n|\leqslant r^p+\frac{(A-B)(p-a)}{(k+1)(1-B)}r^{k+1}$$

and

$$|f(z)|\geqslant r^p-\sum_{n=k+1}^{\infty}|a_n|r^n\geqslant r^p-r^{k+1}\sum_{n=k+1}^{\infty}|a_n|\geqslant r^p-\frac{(A-B)(p-a)}{(k+1)(1-B)}r^{k+1}.$$

Thus (3.1) follows. Further

$$|f'(z)| \leqslant pr^{p-1} + \sum_{n=k+1}^{\infty} n|a_n| r^{n-1} \leqslant pr^{p-1} + r^k \sum_{n=k+1}^{\infty} n|a_n| \tag{3.3}$$

and

$$|f'(z)| \geqslant pr^{p-1} - \sum_{n=k+1}^{\infty} n|a_n| r^{n-1} \geqslant pr^{p-1} - r^k \sum_{n=k+1}^{\infty} n|a_n|. \tag{3.4}$$

But, from Theorem 1, it holds that

$$\sum_{n=k+1}^{\infty} n|a_n| \leqslant \frac{(A-B)(p-a)}{(1-B)}. \tag{3.5}$$

The inequalities in (3.2) follow now by using (3.5) in (3.3) and (3.4).

Equality in (3.1) and (3.2) is obtained if we take

$$f(z) = z^p - \frac{(A-B)(p-a)}{(k+1)(1-B)} z^{k+1}. \tag{3.6}$$

Note For the above function, equality on the left hand side of (3.1) is obtained at $z=r$ whereas on the right hand side equality is obtained at $z=-r$ when $k=p+1, p+3, p+5, \cdots$; $z=ir$ when $k=p+2, p+6, p+10, \cdots$ and $z=re^{i\pi/(k-p)}$ when $k=p+4, p+8, p+12, \cdots$. Similarly, the points where equality holds in (3.2) can be obtained.

Corollary 1 If $f(z) \in P_k(A, B, p, a)$, then the disc U is mapped by $f(z)$ onto a domain that contains the disc

$$|w| < \frac{(k+1)(1-B)-(A-B)(p-a)}{(k+1)(1-B)}.$$

The result is sharp with extremal function $f(z)$ given by (3.6).

The above corollary follows if we let $r \to 1$ in the left hand side inequality in (3.1). An interesting case appears when $A=1$. In this case $|w| < \frac{k+1-p+a}{k+1}$.

4. Integral operators

Theorem 3 Let c be a real number such that $c > -p$. If $f(z) \in P_k(A, B, p, a)$, then the function $F(z)$ defined by

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{4.1}$$

also belongs to $P_k(A, B, p, a)$.

Proof Let $f(z) = z^p - \sum_{n=k+1}^{\infty} |a_n| z^n$. Then from the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=k+1}^{\infty} |b_n| z^n,$$

where

$$|b_n| = \left(\frac{c+p}{c+n}\right) |a_n|.$$

Therefore

$$\sum_{n=k+1}^{\infty} n(1-B)|b_n| = \sum_{n=k+1}^{\infty} n(1-B)\left(\frac{c+p}{c+n}\right)|a_n|$$

$$< \sum_{n=k+1}^{\infty} n(1-B)|a_n| < (A-B)(p-a),$$

since $f(z)\in P_k(A, B, p, a)$. Hence, by Theorem 1, $F(z)\in P_k(A, B, p, a)$.

Theorem 4 Let c be a real number such that $c > -p$. If $F(z)\in P_k(A, B, p, a)$, then the function $f(z)$ defined by (4.1) is p-valent in $|z| < R^*$, where

$$R^* = \inf_{\substack{n>k+1\\ k>p>1}} \left[\left(\frac{c+p}{c+n}\right)\left(\frac{(1-B)n}{(A-B)(p-a)}\right)\right]^{\frac{1}{n-p}}$$

This result is sharp.

Proof Let $F(z) = z^p - \sum_{n=k+1}^{\infty} |a_n| z^n$. It follows then from (4.1) that

$$f(z) = \frac{z^{1-c}}{c+p}[z^c F(z)]' = z^p - \sum_{n=k+1}^{\infty} \left(\frac{c+n}{c+p}\right)|a_n| z^n.$$

In odrer to obtain the required result it sufficies to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p \quad \text{in } |z| < R^*.$$

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=k+1}^{\infty} n\left(\frac{c+n}{c+p}\right)|a_n| z^{n-p}\right| < \sum_{n=k+1}^{\infty} n\left(\frac{c+n}{c+p}\right)|a_n||z|^{n-p}.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p, \quad \text{if} \quad \sum_{n=k+1}^{\infty} n\left(\frac{c+n}{c+p}\right)|a_n||z|^{n-p} < p. \tag{4.2}$$

But Theorem 1 confirms that

$$\sum_{n=k+1}^{\infty} np\left(\frac{1-B}{(A-B)(p-a)}\right)|a_n| < p.$$

Hence (4.2) will be satisfied if

$$n\left(\frac{c+n}{c+p}\right)|a_n||z|^{n-p} < np\left(\frac{1-B}{(A-B)(p-a)}\right)|a_n|, \quad n = k+1, k+2, \cdots, k>p>1$$

or if

$$|z| < \left[\left(\frac{c+p}{c+n}\right)\left(\frac{(1-B)p}{(A-B)(p-a)}\right)\right]^{\frac{1}{n-p}}, \quad n = k+1,\ k+2,\ \cdots, \quad k>p>1.$$

Therefore $f(z)$ is p-valent in $|z| < R^*$. Sharpness follows if we take

$$F(z) = z^p - \frac{(A-B)(p-a)}{n(1-B)} z^n, \quad n>k+1, \quad k>p>1.$$

5. Radius of convexity

Theorem 5 If $f(z)\in P_k(A, B, p, a)$, then $f(z)$ is p-valently convex in the disc $|z|<R$, where

$$R=\inf_{\substack{n\geq k+1,\\ k\geq p\geq 1}}\left[\frac{p^2(1-B)}{n(A-B)(p-a)}\right]^{\frac{1}{n-p}}$$

The result is sharp.

Proof In order to establish the required result it sufficies to show that $\left|1+\frac{zf''(z)}{f'(z)}-p\right|<p$ in $|z|<R$. Let $f(z)=z^p-\sum_{n=k+1}^{\infty}|a_n|z^n$. Then we have

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right|=\left|\frac{-\sum_{n=k+1}^{\infty}n(n-p)|a_n|z^{n-p}}{p-\sum_{n=k+1}^{\infty}n|a_n|z^{n-p}}\right|\leq\frac{\sum_{n=k+1}^{\infty}n(n-p)|a_n||z|^{n-p}}{p-\sum_{n=k+1}^{\infty}n|a_n||z|^{n-p}}.$$

Therefore $\left|1+\frac{zf''(z)}{f'(z)}-p\right|<p$, if

$$\sum_{n=k+1}^{\infty}n(n-p)|a_n||z|^{n-p}<p^2-\sum_{n=k+1}^{\infty}pn|a_n||z|^{n-p}$$

or if

$$\sum_{n=k+1}^{\infty}\left(\frac{n}{p}\right)^2|a_n||z|^{n-p}<1. \tag{5.1}$$

Also, by Theorem 1, we have

$$\sum_{n=k+1}^{\infty}n\left(\frac{(1-B)}{(A-B)(p-a)}\right)|a_n|\leq 1.$$

Hence (5.1) will be satisfied if

$$\frac{n^2}{p^2}|a_n||z|^{n-p}<n\left(\frac{(1-B)}{(A-B)(p-a)}\right)|a_n|,\quad n=k+1,k+2,\cdots,\quad k\geq p\geq 1.$$

or if

$$|z|<\left[\frac{p^2(1-B)}{n(A-B)(p-a)}\right]^{\frac{1}{n-p}},\quad n=k+1,k+2,\cdots,\quad k\geq p\geq 1.$$

Therefore $f(z)$ is p-valently convex in $|z|<R$. Sharpness follows if we take

$$f(z)=z^p-\frac{(A-B)(p-a)}{n(1-B)}z^n,\quad n\geq k+1,\quad k\geq p\geq 1.$$

6. Closure properties

In this section we show that the class $P_k(A, B, p, a)$ is closed under "arithmetic mean" and "convex linear combinations".

Theorem 6 Let $f_j(z)=z^p-\sum_{n=k+1}^{\infty}|a_{nj}|z^n$, $j=1,2,\cdots,m$. If $f_j\in P_k(A, B, p, a)$ for each $j=1.2.\cdots, m$, then the function $h(z)=z^p-\sum_{n=k+1}^{\infty}|b_n|z^n$ also belongs to $P_k(A, B, p, a)$, where $b_n=\frac{1}{m}\sum_{j=1}^{m}a_{nj}$.

Proof Since $f_j(z)\in P_k(A, B, p, a)$, it follows from Theorem 1 that

$$\sum_{n=k+1}^{\infty}n(1-B)|a_{nj}|\leqslant(A-B)(p-a), \quad j=1,2,\cdots, m.$$

Therefore

$$\sum_{n=k+1}^{\infty}n(1-B)|b_n|\leqslant\sum_{n=k+1}^{\infty}[n(1-B)\{\frac{1}{m}\sum_{j=1}^{m}|a_{nj}|\}]\leqslant(A-B)(p-a).$$

Hence, by Theorem 1, $h(z)\in P_k(A, B, p, a)$.

Theorem 7 Let

$$f_p(z)=z^p, \quad f_n(z)=z^p-\frac{(A-B)(p-a)}{n(1-B)}z^n \quad (n\geqslant k+1, k\geqslant p\geqslant 1).$$

Then $f(z)\in P_k(A, B, p, a)$ if and only if it can be expressed in the form

$$f(z)=\lambda_p f_p(z)+\sum_{n=k+1}^{\infty}\lambda_n f_n(z),$$

where $\lambda_n\geqslant 0$ and $\lambda_p+\sum_{n=k+1}^{\infty}\lambda_n=1$.

Proof Let us assume that

$$f(z)=\lambda_p f_p(z)+\sum_{n=k+1}^{\infty}\lambda_n f_n(z)=z^p-\sum_{n=k+1}^{\infty}\frac{(A-B)(p-a)}{n(1-B)}\lambda_n z^n.$$

Then

$$\sum_{n=k+1}^{\infty}[\{n(1-B)\}\{\frac{(A-B)(p-a)}{n(1-B)}\}\lambda_n]=(A-B)(p-a)\sum_{n=k+1}^{\infty}\lambda_n\leqslant(A-B)(p-a).$$

Hence, by Theorem 1, $f(z)\in P_k(A, B, p, a)$.

Conversely, let $f(z)\in P_k(A, B, p, a)$. It follows then from Theorem 1 that

$$|a_n|\leqslant\frac{(A-B)(p-a)}{n(1-B)} \quad (n=k+1, k+2, \cdots, k\geqslant p\geqslant 1).$$

Setting

$$\lambda_n=\frac{n(1-B)}{(A-B)(p-a)}|a_n|, \quad (n=k+1, k+2, \cdots, \quad k\geqslant p\geqslant 1),$$

and

$$\lambda_p=1-\sum_{n=k+1}^{\infty}\lambda_n \quad ,$$

we have

$$f(z)=\lambda_p f_p(z)+\sum_{n=k+1}^{\infty}\lambda_n f_n(z).$$

This completes the proof of the theorem.

Our results generalize the results of Kumar [3]. Gupta and Jain [2], Shukla and Dashrath [4] and Aouf [1].

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