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Prime Cycle-Factorizations of Complete Graphs*

Wang Jianfang

(Institute of Applied Mathematics, Academia Sinica. Beijing)

Wang Fuda

(Chinese College of Coal Economic, Yantai)

§ I. Introduction

C_k denote a cycle of length k. A factor H of a graph G is called $\{C_{k_1}, C_{k_2}, \cdots, C_{k_t}\}$-factor if each component of H is one of $\{C_{k_1}, C_{k_2}, \cdots, C_{k_t}\}$, A $\{C_{k_1}, C_{k_2}, \cdots, C_{k_t}\}$-factorization of G is a partition $\{E_1, E_2, \cdots, E_m\}$ of $E(G)$ such that each spanning subgraph (V, E_i) is a $\{C_{k_1}, C_{k_2}, \cdots, C_{k_t}\}$-factor. If $t=1$, $\{C_{k_1}\}$-factor write simply C_{k_1}-factor. Enomoto ect. [1] discussed C_k-factorization of complete bipartite graphs. In this paper, we shall discuss cycle-factorization of complete graphs. A cycle with prime length is called prime cycle. If there exists a $\{C_{k_1}, C_{k_2}, \cdots, C_{k_t}\}$-factorization of G then we say that G is $\{C_{k_1}, C_{k_2}, \cdots, C_{k_t}\}$-facatorizable.

Let ϕ be a permutation of symmetric group S_n and ϕ' the permutation for pair group of S_n defined $\phi'\{i,j\} = \{\phi_i, \phi_j\}$ The ϕ' is called the induced permutation of ϕ.

Let G_1 and G_2 be two given graphs. Their cartesian product, wite $G_1 \times G_2$, is defined that

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

and

$$E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1) \text{ and } v_1 = v_2, \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)\}.$$

Their lexicographic product, write $G_1 \otimes G_2$ is defined that

$$V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$$

and

$$E(G_1 \otimes G_2) = \{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1) \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(G)\}$$

I_n denote isolate graph of order n. The terminologies and notations using in this paper are same in [2], [3].

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§ 2 Theorems and Proof

Lemma 1 If n is odd, then there exist C_n- factorization of K_n. This result can be found in [2]. Here we give a decomposition. Let

$$V = V(K_n) = \{a_1, a_2, \cdots, a_n\}$$

$$\phi = (a_1, a_2, \cdots, a_{n-1})(a_n).$$

$$C = (a_1, a_{n-1}, a_2, a_{n-2}, \cdots, a_{(n-1)/2}a_{(n-1)/2+1}, \cdots, a_n, a_1).$$

$$E_0 = E(C).$$

then $\{(V, (\phi')^i(E_0) | i = 0, 1, \cdots, (n-1)/2 - 1\}$ form a C_n-factorization of K_n.

Lemma 2 $K_{nm} = K_n \otimes I_m \bigcup I_n \times K_n$ and

$$E(K_n \otimes I_m) \bigcap E(I_n \times K_n) = \phi$$

Proof It immediately follows from definition.

Lemma 3

$$K_{n_1, n_2, \cdots, n_i} = K_{n_1, n_2, \cdots, n_i} \bigcup_{i=2}^{i-1} I_{n_1, n_2, \cdots, n_{i_i}} \times (K_{n_i} \otimes I_{n_{i+1}, \cdots, n_t}) \bigcup I_{n_1, n_2, \cdots, n_{i-1}} \times K_{n_i}$$

and in this decomposition each other has no edge in common.

Proof Repeat to use lemma 2 we can obtain the lemma.

Lemma 4 If m is odd, then there exists a C_n-factorization of $C_n(m) = C_n \otimes I_m$.

Proof

$$A^i = \{a_1^i, a_2^i, \cdots, a_n^i), \ i = 1, 2, \cdots, m$$

Let

$$\phi_j = (a_j^1 a_j^2 \cdots a_j^m)$$
$$(a_1^1) \cdots (a_{j-1}^1)(a_{j+1}^1) \cdots (a_n^1) \cdots$$
$$(a_1^i) \cdots (a_{j-1}^i)(a_{j+1}^i) \cdots (a_n^i) \cdots$$
$$(a_1^m) \cdots (a_{j-1}^m)(a_{j+1}^m) \cdots (a_n^m), \ j = 1, 2, \cdots, n.$$

$$\phi = (a_1^1, a_1^2, \cdots, a_1^m)(a_2^1, a_2^2, \cdots, a_2^m) \cdots (a_n^1, a_n^2, \cdots, a_n^m).$$

It is obvious that $C_n(m)[A^i] \cong C_n$. We only consider that n is odd, for n even the proof is similar. Let

$$E' = \bigcup_{i=0}^{m-1} (\phi')^i (E(C_n(m)[A'])) $$

$$E_i = (\phi_2' \phi_4' \cdots \phi_{n-1}' (\phi_n^{-1})')^i (E'), \ \ i = 1, 2, \cdots, m-1$$

It is easy to see that $C_n(m)[E_i]$ is a C_n-factor of $C_n(m)$. It is noly need to prove that $\{E_i | i = 0, 1, \cdots, m-1\}$ form a partition of $E(C_n(m))$. For this it is need to prove that

(Ⅰ) If $i \neq j$, then $E_i \cap E_j = \phi$ and

(Ⅱ)

$$|E(C_n(m))| = \sum_{i=0}^{m-1} |E_i|$$

We first prove (Ⅰ). Now we prove that there is no edge with form (a_{n-1}^p, a_n^q) in E_i and E_j in common. Supose otherwise, there exists such an edge $(a_{n-1}^p, a_n^q) \in E_i \cap E_j$. Then there exist t_1 and t_2. such that

$$p \equiv t_1 + i \pmod{m},$$
$$p \equiv t_2 + j \pmod{m},$$
$$q \equiv t_1 + i \pmod{m},$$
$$q \equiv t_2 + j \pmod{m},$$
$$t_1 + i \equiv t_2 + j \pmod{m}, \qquad (1)$$
$$t_1 - i \equiv t_2 - j \pmod{m}, \qquad (2)$$
$$2i \equiv 2j \pmod{m}$$

Since n is odd, $(2, m) = 1$. This $i \equiv j \pmod{m}$ to see [4]. Because $i, j < m$, therefor $i = j$.

It contrary to the supose $i \neq j$. Hence there is no edge with form (a_{n-1}^p, a_n^q) belong to either both E_i and E_j. It is obvious that there is no an edge with other from belong to $E_i \cap E_j$.

Now we prove (Ⅱ). Since $C_n(m)[E_i]$ is a C_n-factor of $C_n(m)$, it contains in components one of which is isomorphic to C_n. Hence

$$|E_i| = nm, \quad \sum_{i=0}^{m-1} |E_i| = mnm = nm^2 .$$

Obviously $|E(C_n(m))| = nm^2$. Thus

$$|E(C_n(m))| = \sum_{i=0}^{m-1} |E_i|.$$

From above we know that $\{E_0, E_1, \cdots, E_{m-1}\}$ is a partition of $E(C_n(m))$.

Hence $\{(V(C_n(m)), E_i) \mid i = 0, 1, \cdots, m-1\}$ forms a C_n-factorization of $C_n(m)$.

Theorem 5 If m, n are odd numbers, then $K_n(m) = K_n \otimes I_m$ is C_n-factorizable.

Proof Arccoding to lemma 1, K_n is C_n-factorizable. Each factor of the C_n-factorization is isomorphic to C_n. The C_n-factorization has $(n-1)/2$ such factors, write these factors as $C^1, C^2, \cdots, C^{(n-1)/2}$, $E(C^i) \cap E(C^j) = \phi$ if $i \neq j$. Thus we get a decomposition of $K_n(m)$ as

$$K_n(m) = K_n \otimes I_m = \left(\bigcup_{i=1}^{(n-1)/2} C^i\right) \otimes I_m .$$

Since $C^i \cong C^n$, $C^i \otimes I_m$ is C_n-factorizable by lemma 4. Thus $K_n(m)$ is C_n-factorible.

Lemma 6 If G is C_n-factorizable, then for any positive integer m, $I_m \times G$ is C_n-factorizable.

Proof It is obvious.

Theorem 7 If n is odd. Then for any positive integer m, K_{n^m} is C_n-factorizable.

Proor By Lemma 3, K_{n^m} can be decomposed as

$$K_{n^m}=K_n\otimes I_{n^{m-1}}\cup\bigcup_{i=2}^{m-1}I_{n^{i-1}}\times(K_n\otimes I_{n^{m-i}})\cup I_{n^{m-1}}\times K_n.$$

Because n is odd, for any $j\geqslant 1$, n^j is odd. According to Theorem 5 and Theorem 6, each factor in the above decomposition of K_{n^m} is C_n-factorizable. Thus K_{n^m} is C_n-factorizable.

Theorem 8 If $p\geqslant 3$ is odd. Then K_p is prime cycle-factorizable. That is, there exists a cycle-factorization each factor of which is prime length cycle-factor.

Proof Give prime factorization of P. write

$$P=n_1^{m_1}n_2^{m_2}\cdots n_t^{m_t}.$$

each n_i is prime number. Because P is odd, each n_i is odd.

We first give decomposition K_p according to Lemma 3.

$$\begin{aligned}K_p=&K_{n_1^{m_1}}\otimes I_{n_2^{m_2}\cdots n_t^{m_t}}\\&+\cup\Big(\bigcup_{i=2}^{t-2}I_{n_1^{m_1}\cdots n_{i-1}^{m_{i-1}}}\times(K_{n_i^{m_i}}\otimes I_{n_{i+1}^{m_{i+1}}\cdots n_t^{m_t}})\Big)\\&+\cup I_{n_1^{m_1}\cdots n_{t-1}^{m_{t-1}}}\times K_{n_t^{m_t}}.\end{aligned}$$

By theorem 7, theorem 5 and theorem 6, we know that for any i, $1\leqslant i<t$,

$$I_{n_1^{m_1}\cdots n_{i-1}^{m_{i-1}}}\times(K_{n_i^{m_i}}\otimes I_{n_{i+1}^{m_{i+1}}\cdots n_t^{m_t}})$$

is C_{n_i}-factorizable.

$$K_{n_1^{m_1}}\otimes I_{n_1^{m_1}\cdots n_t^{m_t}}$$

is C_{n_1}-factorizable.

$$I_{n_1^{m_1}\cdots n_{t-1}^{m_{t-1}}}\times K_{n_t^{m_t}}$$

is C_{n_t}-factorizable.

The proof is completed.

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完全图的素圈因子分解

王 建 方　　　　王 福 达

（中国科学院应用数学研究所，北京）（中国煤炭经济学院，烟台）

摘 要

图的因子理论是图论研究中最活跃的课题之一．其中心问题是把一个图分解成具有给定性质的因子．

设 $G_1, G_2, \cdots, G_k$ 是一组给定的非空图．图 H 被称为图 G 的一个 $\{G_1, G_2, \cdots, G_k\}$－因子，如果 H 是 G 的一个支撑子图并且 H 的每个分枝同构于 $\{G_1, G_2, \cdots, G_k\}$ 中的一个图．若 $k=1$，$\{G_1\}$－因子简称为 G_1－因子．图 $G=(V, E)$ 的一个 $\{G_1, \cdots, G_k\}$－因子分解是边集 E 的一个分划 $\{E_1, E_2, \cdots, E_k\}$，使得每个 (V, E_i) 都是一个 $\{G_1, G_2, \cdots, G_k\}$－因子．统称这类因子和因子分解为分枝因子和分枝因子分解．若 $G_1, G_2, \cdots, G_k$ 都是圈，则特别称相应的因子和因子分解为圈因子和圈因子分解．[1][2]研究了一些类型的分枝因子的存在性．本文将研究图的圈因子分解．

设 ϕ 是对称群 S_n 的一个置换，ϕ' 表示由 ϕ 导出的对称群 S_n 的配对群（参看[3]）的置换，满足 $\phi'\{i, j\}=\{\phi i, \phi j\}$．

我们称长度为素数的圈为素圈．

对给定的两个图 G_1, G_2，$G_1 \times G_2$ 表示 G_1 称 G_2 的迪卡尔乘积圈，定义如下：

$$V(G_1 \times G_2)=V(G_1) \times V(G_2).$$

$$E(G_1 \times G_2)=\{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1) \text{ 且 } v_1=v_2, \text{ 或 } u_1=u_2 \text{ 且 } v_1v_2 \in E(G_2)\}.$$

G_1 和 G_2 的逻辑积，记为 $G_1 \otimes G_2$，定义为：

$$V(G_1 \otimes G_2)=V(G_1) \times V(G_2),$$

$$E(G_1 \otimes G_2)=\{(u_1, v_1)(u_2, v_2) \mid u_1u_2 \in E(G_1) \text{ 或 } u_1=u_2 \text{ 且 } v_1v_2 \in E(G_2)\}.$$

I_n 表示具有 n 个顶点的孤立图．