Journal of Mathematical Research and Exposition
Vol. 10, No.2, May, 1990

Maximal Operators and Hilbert Transforms Along Convex Curves*

Gao Fuchang

(Institute of mathmatics, Fudan University)

1. Introduction

Let $\Gamma(t)=(y_1(t), y_2(t), \cdots, y_n(t))$ be a continuous curve in R^n, define the "maximal function along Γ" of f by

$$Mf(x)=\sup_{h>0}\left|\frac{1}{h}\int_0^h f(x-\Gamma(t))dt\right| \quad (1)$$

and the "Hilbert transform along Γ" of f by

$$Hf(x)=P.V.\int_{-\infty}^{+\infty} f(x-\Gamma(t))\frac{dt}{t} \quad (2)$$

It is of substantial interest to seek condition on Γ along which we have

$$\|Mf\|_p \leqslant C_p\|f\|_p \quad (3)$$

$$\|Hf\|_p \leqslant C_p\|f\|_p \quad (4)$$

for all $f\in L^p(R^n)$.

E. M. Stein and S. Wainger proved in [1] that if Γ is a homogeneous curve, then (3) and (4) hold for all p, $1<p<\infty$. D. Weinberg extended this result to approximately homogeneous curves, see [2]. A little later, W. C. Nestlerode considered odd curve, and showed in [3] that for highly monotone curves, (3) holds for $2\leqslant p\leqslant\infty$, and (4) holds for $p=2$. Recent results are found in [4]. In case $n=2$, better results are obtained for convex curves, see for example [5] and [6].

The main object of this paper is to extend the result of [3] to all p, $1<p<\infty$, under a condition which is weaker than the one given out in [3].

2. Statement of the Current Result

We shall consider curves $\Gamma=(y_1, y_2, \cdots, y_n): R\to R^n$ in which we assume the coordinate function y_j satisfy

$$y_j: R\to R \text{ is either even or odd and of class } C^n(R).$$

* Received Nov. 26, 1988

$$\gamma_j(0)=0,\ \gamma_j(t)>0,\ \text{for } t>0,\ (1\leqslant j\leqslant n),\ \text{and } \gamma_1(t)=t,\ (t>0). \qquad (5)$$

Just as in [3], We inductively define the differential operators along Γ by

$$D^1 f=f',\ D^{k+1}=(D^k f/D^k\gamma_k)' \qquad k=1,2,3,\cdots. \qquad (6)$$

We say Γ is convex, if

$$D^k\gamma_k(t)>0,\ \text{for } t>0,\quad 1\leqslant k\leqslant n. \qquad (7)$$

For convenience of the statement, we introduce some terms: By a function f "is quasi-increasing", we mean there exist $C_1>0$, $C_2>0$, such that $f(x)>C_2 f(t)$ for $t<x<C_1 t$; By a function g "has b.d.t. property", we mean that there exists a constant $\lambda>1$, such that for $C\geqslant\lambda$, $g(Ct)\geqslant 2g(t)$. Our main result is the following.

Theorem A Let $\Gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ satisfies the basic assumption (5) and the "convexity" assumption (7). If Γ also satisfies

(a) $|D^k\gamma_j/D^k\gamma_k|$ have b.d.t. property for $1\leqslant k<j\leqslant n$;

(b) $D^1\gamma_1, D^2\gamma_2,\cdots,D^{n-1}\gamma_{n-1}, D^n\gamma_n$ are quasi-increasing,

then we have

$$\|Mf\|_p\leqslant C_p\|f\|_p \qquad (1<p\leqslant\infty) \qquad (8)$$

$$\|Hf\|_p\leqslant C_p\|f\|_p \qquad (1<p<\infty) \qquad (9)$$

3. Main Idea of the Proof

By inductive method, we consider the case $n=m$. Put

$$M_k f(x)=\frac{1}{|I_k|}\int_{I_k} f(x_1-\gamma_1(t),\cdots,x_m-\gamma_m(t))\,dt \qquad (10)$$

$$N_k f(x)=\frac{1}{|I_k|}\int_{I_k}\frac{1}{\gamma_m(\lambda^k)}\int_0^{\gamma_m(\lambda^k)} f(x_1-\gamma_1(t),\cdots,x_{m-1}-\gamma_{m-1}(t),x_m-s)\,ds\,dt \qquad (11)$$

where $I_k=(\lambda^k,\lambda^{k+1})$

$$\lambda=\inf\{a>0;\ D^k\gamma_j(\beta t)/D^k\gamma_k(\beta t)\geqslant 2D^k\gamma_j(t)/D^k\gamma_k(t),\ \gamma_j(\beta t)>2\gamma_j(t)\ \ \forall\beta>a,\ t>0,\ 1\leqslant k<j\leqslant n\}$$

The hardnet is the estimation of $\|\sup_k|(M_k-N_k)|f\|_p$. To do this we introduce a Paley-Littlewood decomposition $R_k\{(\xi_1,\xi_2,\cdots,\xi_n)\in R^n;\ \gamma_n(\lambda^k)\leqslant|\xi_n|<\gamma_n(\lambda^{k+1})\}$. Choose $w_k\in C_0(R^n-\{0\})$ such that $0\leqslant w_k(\xi)\leqslant 1$, Supp $w_k\subset R_{k+1}\cup R_k\cup R_{k-1}$, and $\sum_k w_k=1$. Define operators $\widehat{T_k f}(\xi)=w_k(\xi)\hat{f}(\xi)$, we have

$$\sup_k|(M_k-N_k)f|\leqslant\sum_j\Big(\sum_k|(M_k-N_k)T_{j+k}f|^2\Big)^{\frac{1}{2}} \qquad (12)$$

An application of plancherel formular gives

$$\|\sup_k|(M_k-N_k)f|\|_2\leqslant C\sum_j\|2^{-|j|/m}\hat{f}\|_2\leqslant C\|f\|_2 \qquad (13)$$

By interpolation and duality, we can extend (13) to all p, $1<p\leqslant\infty$. (see [7]

for the first step).

(9) is proved in a similar way

4. Remark

Our "convexity" assumption is equivilant to that in [4], since by triangularization, we can see that the matrix

$$\begin{bmatrix} y_1'(t) & y_2'(t) & \cdots & y_m'(t) \\ y_1''(t) & y_2''(t) & \cdots & y_m''(t) \\ \cdots & \cdots & & \cdots \\ y_1^{(m)}(t) & y_2^{(m)}(t) & \cdots & y_m^{(m)}(t) \end{bmatrix}$$

is equiviant to matrix

$$\begin{bmatrix} D^1y_1(t) & D^1y_2(t) & \cdots & D^1y_m(t) \\ 0 & D^1y_1(t)D^2y_2(t) & \cdots & D^1y_1(t)D^2y_m(t) \\ \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & D^1y_1(t)D^2y_2(t)\cdots D^my_m(t) \end{bmatrix}$$

References

[1] E. M. Stein & S. Wainger, Problems in harmonic analysis related to curvature, Bull. Amer. Math. Soc. 84 (1978).

[2] D. Weinberg, The Hilbert transforms and Maximal functions for approximately nomogeneous curves, Trans, Amer, Math, Soc. 267 (1981), 295—306.

[3] W. C. Nes lerode, Singular integrals and Maximal functions associated with highly monotone curves, Trans. Amer. Math. Soc. 267 (1981), 435—444.

[4] A. Nagel, J. Vance, S. Wainberg, The Hilbert transforms for convex curves in R^n, Amer. J. Math. 108 (1986), —71.

[5] A. Cordoba, A. Nagel, S. Wainger, D. Weinberg, L^p bounds for Hilbert transforms along curves, Iven. Math., 83 (1986), 59—71.

[6] H. Carlsson, M. Christ, A. Cordoba, J. Duoandikoetxea, J. L. Rubio de Francia, J, Vance, S. Wainger, D. Weinberg, L^p estimates for Maximal functions and Hibert transforms along flat convex curves in R^n, Bull. Amer. Math. Soc., 14 (1986), 263—267.

[7] A. Nagel, E. M. Stein, S. Wainger, Differentiation in lacunary directions, Proc. Natl. Acad. Sci. USA. 75 (1978), 1060—1062.

凸曲线上极大函数与Hilbert变换

高　福　昌

(复旦大学数学研究所，上海)

摘　　要

本文给出了沿 R^n 中凸曲线的极大函数算子与Hilbert变换具有 L^p $(1<p<\infty)$ 有界性的一个充分条件；当 $p=2$ 时，本文给出了另一充分条件，它在某种意义下是必要的．