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Medium Algebras*

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Abstract

In this paper, a new type algebra which is so-called Medium algebra is introduced. The equational class M of all Medium algebras is characterized. And the relationships between Medium algebras and other algebras are studied.

1. Introduction

Fuzzy mathematics which was initialed by Zadeh [4] in 1965, has been rapidly developed with manyfold applications ranging from engineering and computer science to medical diagnosis and social behavior. The basis of fuzzy mathematics, however, had not been well established until Zhu and Xiao [3] introduced the Medium logic in recent years. Medium algebra is the algebraic abstract of MP system in Medium logic just as Boolean algebra abstracting the two valued Propositional Calculus. The main purpose of this paper is to investigate the various properties of Medium algebras.

2. Definitions and Basic Properties

Definition 2.1 A Medium algebra is an algebra $\langle M, +, \cdot, ^-, \sim, 0, 1\rangle$ of type $\langle 2,2,1,1,0,0\rangle$ whose reduct $\langle M, +, \cdot, ^-, 0, 1\rangle$ is a De Morgan algebra and such that for all $x, y \in M$,

(1) $\overline{\tilde{x}} = \tilde{x}$

(2) $\tilde{\tilde{x}} = x + \bar{x}$

(3) $x\bar{x} + \tilde{x} = \tilde{x}$

(4) $\widetilde{x+y} = \tilde{x}\,\tilde{y} + \tilde{x}\,\bar{y} + \bar{x}\,\tilde{y}$

(5) $\widetilde{xy} = \tilde{x}\,\tilde{y} + \tilde{x}\,y + x\,\tilde{y}$

(6) $x + \bar{x} + \tilde{x} = 1$.

Obviously, the class M of all Medium algebras is an equational class.

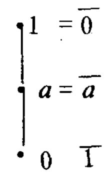

Figure 1.

Example 2.1 Let $B = \langle B, +, \cdot, -, 0, 1\rangle$ be a Boolean algebra. Define "$\sim$" by setting $\tilde{x} = 1$ for all $x \in B$. Then $\langle B, +, \cdot, -, \sim, 0, 1\rangle$ is a Medium algebra.

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Example 2.2 Let $M=M_1$ whose Hasse diagram is depicted in Figure 1. Define "$\sim$" by setting $\tilde{a}=1$, $\tilde{0}=\tilde{1}=a$. Then $\langle M, +, \cdot, -, \sim, 0, 1\rangle$ is a Medium algebra which is called the standard Medium algebra.

Theorem 2.1 Let $M=\langle M, +, \cdot, -, \sim, 0, 1\rangle$ be a Medium algebra. Then

(1) $\tilde{\tilde{\tilde{x}}}=\tilde{x}$

(2) $\tilde{x}\,\tilde{y}=\widetilde{xy}\,\widetilde{x+y}$

(3) $\tilde{x}+\tilde{y}=\widetilde{xy}+\widetilde{x+y}$

(4) $\tilde{x}\tilde{\tilde{x}}=\tilde{0}=\tilde{1}$

(5) $\tilde{\bar{x}}\leqslant\tilde{x}\,\tilde{\tilde{x}}$, i.e. $\tilde{\bar{x}}\leqslant\tilde{0}$

(6) If $x\leqslant y\leqslant\tilde{0}$ or $x\geqslant y\geqslant\tilde{0}$ then $\tilde{x}\leqslant\tilde{y}$.

Proof (1) $\tilde{\tilde{\tilde{x}}}=\widetilde{\widetilde{x+\bar{x}}}=\tilde{x}\,\tilde{\tilde{x}}+\tilde{x}\,\bar{\bar{x}}+\bar{x}\,\tilde{\bar{x}}=\tilde{x}$. (2) $\widetilde{xy}\,\widetilde{x+y}=(\tilde{x}\,\tilde{y}+x\tilde{y}+\tilde{x}y)(\tilde{x}\,\tilde{y}+\bar{x}\tilde{y}+\tilde{x}\bar{y})=\tilde{x}\,\tilde{y}+\tilde{x}y\bar{y}+x\bar{x}\tilde{y}=\tilde{x}\,\tilde{y}$. (3) $\widetilde{x+y}+\widetilde{xy}=\tilde{x}\,\tilde{y}+\tilde{x}\bar{y}+\bar{x}\tilde{y}+\tilde{x}\,\tilde{y}+\tilde{x}y+x\tilde{y}=\tilde{x}(y+\bar{y}+\tilde{y})+\tilde{y}(x+\bar{x}+\tilde{x})=\tilde{x}+\tilde{y}$. (4) Since $\tilde{x}+\tilde{\tilde{x}}=1$, we have $\widetilde{\tilde{x}+\tilde{\tilde{x}}}=\tilde{1}=\tilde{1}=\tilde{0}$. Thus $\tilde{\tilde{x}}\,\tilde{\tilde{\tilde{x}}}+\tilde{\tilde{x}}\,\bar{\tilde{\tilde{x}}}+\tilde{\bar{x}}\,\tilde{\tilde{x}}=\tilde{1}$. $\tilde{\tilde{x}}\,\bar{\tilde{\tilde{x}}}\leqslant\tilde{\tilde{\tilde{x}}}=\tilde{x}$ implies $\tilde{\tilde{x}}\,\bar{\tilde{\tilde{x}}}\leqslant\tilde{x}\,\tilde{\tilde{x}}$. Similarly, $\tilde{x}\,\tilde{\bar{x}}\leqslant\tilde{\tilde{x}}$ implies $\tilde{x}\,\tilde{\bar{x}}\leqslant\tilde{x}\,\tilde{\tilde{x}}$. Therefore $\tilde{x}\,\tilde{\tilde{x}}=\tilde{1}=\tilde{0}$. (5) $x\tilde{x}\leqslant\tilde{x}$ implies $\bar{\tilde{x}}\leqslant\overline{x\tilde{x}}=\bar{x}+x=\tilde{\tilde{x}}$. Since $x\bar{x}=\tilde{\tilde{x}},\tilde{x}\,\bar{\tilde{x}}=\tilde{\tilde{\tilde{x}}}=\bar{\tilde{x}}$, we have $\bar{\tilde{x}}\leqslant\tilde{x}$. Therefore $\bar{\tilde{x}}\leqslant\tilde{x}\,\tilde{\tilde{x}}=\tilde{1}=\tilde{0}$. (6) If $x\leqslant y\leqslant\tilde{0}$, then $xy=x$, we have $\tilde{x}=\widetilde{xy}=\tilde{x}\,\tilde{y}+x\tilde{y}+\tilde{x}y$. Since $\tilde{x}\geqslant\tilde{x}\,\tilde{\tilde{x}}=\tilde{0}\geqslant y\geqslant x$, $\tilde{x}\,\tilde{y}\geqslant x\tilde{y}$. Similarly, $\tilde{x}\,\tilde{y}\geqslant\tilde{x}\,y$. Hence $\tilde{x}=\tilde{x}\,\tilde{y}$. i.e. $\tilde{x}\leqslant\tilde{y}$. It is similar to prove the second assertion.

Suppose S is a nonempty subset of a Medium algebra M, $[S]$ denotes the Medium algebra generated by S, i.e. $[S]$ is the smallest Medium algebra that contains S. Such $[S]$ is characterized by the following theorem.

Theorem 2.2 Let S be a nonempty subset of a Medium algebra M. Then

$$[S]=\{\sum_{i=1}^{n}\pi T_i \mid T_i \Subset S\cup\bar{S}\cup\tilde{S}\cup\tilde{\bar{S}},\ n\geqslant 1\} \qquad (1)$$

where $A\Subset B$ denotes that A is a finite nonempty subset of B, $\bar{S}=\{\bar{x}\mid x\in S\}$, $\tilde{S}=\{\tilde{x}\mid x\in S\}$.

Proof Let A equal to the right of (1). For any $x\in M$, setting $T_1=\{x\}$, $T_2=\{\bar{x}\}$, $T_3=\{\tilde{x}\}$, we have $1=x+\bar{x}+\tilde{x}=\sum_{i=1}^{3}\pi T_i\in A$. Setting $T_1=\{x, \bar{x}, \bar{\tilde{x}}\}$. We have $x\tilde{x}\,\bar{\tilde{x}}=\overline{x+\bar{x}+\tilde{x}}=\bar{1}=0=\sum\pi T_4\in A$.

For any $T_i, S_j\Subset S\cup\bar{S}\cup\tilde{S}\cup\bar{\tilde{S}}$, $(i=1,\cdots,n,\ j=1,\cdots,m)$. It is easy to show that $\sum_{i=1}^{n}\pi T_i+\sum_{j=1}^{m}\pi S_j\in A$, $\sum_{i=1}^{n}\pi T_i\sum_{j=1}^{n}\pi S_j\in A$ and $\overline{\sum_{i=1}^{n}\pi T_i}=\prod_{i=1}^{n}\sum\bar{T}_i\in A$. $\widetilde{\sum_{i=1}^{n}\pi T_i}\in A$ follows immediately by induction. Therefore A is a Medium algebra. If $M'\supseteq S$ is a Medium algebra, then $M'\supseteq S\cup\bar{S}\cup\tilde{S}\cup\tilde{\bar{S}}$. Thus $A\subseteq M'$. That completes the proof.

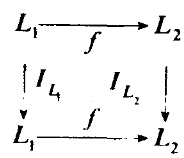
Theorem 2.3 Let $M=\langle M, +, \cdot, -, \sim, 0, 1\rangle$ be a Medium algebra. Then θ

is a congruence relation on $\langle M, +, \cdot, -, ', \sim, 0, 1\rangle$ if and only if θ is a congruence relation on $\langle M, +, \cdot, \sim, 0, 1\rangle$.

Proof We only prove that $(x, y)\in\theta$ implies $(\tilde{x}, \tilde{y})\in\theta$ for all $x, y\in M$. Suppose $(x, y)\in\theta$, we have $(\bar{x}, \bar{y})\in\theta$, $(\tilde{\tilde{x}}, \tilde{\tilde{y}}) = (x+\overline{x}, y+\overline{y})\in\theta$. Since $(\tilde{x}, \tilde{x})\in\theta$ and $(\tilde{y}, \tilde{y})\in\theta$, $(\tilde{x}\tilde{\tilde{x}}, \tilde{x}\tilde{\tilde{y}})\in\theta$ and $(\tilde{y}\tilde{\tilde{x}}, \tilde{y}\tilde{\tilde{y}})\in\theta$. It follows that $(\tilde{x}\tilde{\tilde{y}}, \tilde{0})\in\theta$, $(\tilde{y}\tilde{\tilde{x}}, 0)\in\theta$. Thus $(\tilde{x}\tilde{\tilde{y}}, \tilde{y}\tilde{\tilde{x}})\in\theta$. Therefore $(\tilde{x}, \tilde{y}) = (\tilde{x}(\tilde{y}+\tilde{\tilde{y}}), \tilde{y}(\tilde{x}+\tilde{\tilde{x}})) \doteq (\tilde{x}\tilde{y}+\tilde{x}\tilde{\tilde{y}}, \tilde{x}\tilde{y}+\tilde{y}\tilde{\tilde{x}}) \in\theta$.

3. Coproducts of Medium Algebras

For a lattice L, the elements of $\breve{L}$ are the same as the elements of L but $a<b$ in $\breve{L}$ if and only if $a>b$ in L. The map $I_L: L\to\breve{L}$ is defined by $I_L(a) = a$ for $a\in L$. Note that $a<b \Leftrightarrow I_L(a) > I_L(b)$ for $a, b\in L$. If $f: L_1\to L_2$ is a homomorphism between lattices then $\breve{f}: \breve{L}_1\to\breve{L}_2$ is defined by $\breve{f}(I_{L_1}(a)) = I_{L_2}(f(a))$ for $a\in L_1$. Obviously, $\breve{f}$ is a lattice homomorphism and the diagram commutes.



Note that if $f_1: L_1\to L_2$, $f_2: L_2\to L_3$ are lattice homomorphisms, then $\breve{(f_2\circ f_1)} = \breve{f}_2\circ\breve{f}_1$.

Lemma 3.1 If M is a Medium algebra, then $\breve{M}$ can be made into a Medium algebra by defining $\overline{I_M(x)} = I_M(\bar{x})$, $\widetilde{I_M(x)} = I_M(\tilde{x})$.

Proof Straightforward.

Theorem 3.1 Let $f_1: M_1\to M_2$ be a homomorphism between Medium algebras. Then $\breve{f}_1: \breve{M}_1\to\breve{M}_2$ is also a homomorphism. Moreover, if $f_2: M_2\to M_3$ is another homomorphism between Medium algebras then $\breve{(f_2\circ f_1)} = \breve{f}_2\circ\breve{f}_1$.

Proof Trivial.

Theorem 3.2 Let $(M_s)_{s\in S}$ be a family of Medium algebra, $M\in\mathbf{M}$ and let $(j_s: M_s\to M)_{s\in S}$ be a coproduct of $(M_s)_{s\in S}$. Then $(\breve{j}_s: \breve{M}_s\to\breve{M})_{s\in S}$ is a coproduct of $(\breve{M}_s)_{s\in S}$.

Proof It follows from Lemma 3.1 that $(\breve{M}_s)_{s\in S}$ is a family of Medium algebras. Suppose $(f_s: \breve{M}_s\to L)_{s\in S}$ be a family of homomorphisms between Medium algebras. By Theorem 3.1 $(\breve{f}_s: M_s\to\breve{L})_{s\in S}$ is also a family of homomorphisms between Medium algebras. Since $(j_s: M_s\to M)_{s\in S}$ is a coproduct of $(M_s)_{s\in S}$, there exists a unique homomorphism $f: M\to\breve{L}$ such that $f\circ j_s = \breve{f}_s$ for all $s\in S$. Thus $\breve{f}\circ\breve{j}_s = \breve{(f\circ j_s)} = f_s$, for all $s\in S$. Therefore $(\breve{j}_s: \breve{M}_s\to\breve{M})$ is a coproduct of $(\breve{M}_s)_{s\in S}$.

4. The Equational Class of Medium Algebras

The following example shows that the equational class of Medium algebras $\mathbf{M}$ is a equational proper subclass of the equational class of De Morgan algebras $\mathbf{DM}$.

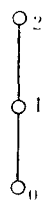
Example 4.1 $M=\langle[0,1],\vee,\wedge,-,0,1\rangle$ is a De Morgan algebra, where $\vee=\max$, $\wedge=\min$, $\bar{x}=1-x$, for all $x\in[0,1]$. Then we can not make M into a Medium algebra.

Proof Suppose there is a unary operation "$\sim$" on $[0,1]$ such that $M=\langle[0,1]\vee,\wedge,-,\sim,0,1\rangle$ is a Medium algebra. Then for $x\in[0,1]$, we have $x\vee\bar{x}\vee\tilde{x}=1$. Thus for $x\neq 0$ or 1, we have $\tilde{x}=1$ and $\tilde{\bar{x}}=\tilde{1}$. i.e. $x\vee\bar{x}=\tilde{1}$. But if $x=0.7\in[0,1]$; $y=0.5\in[0,1]$ $x\vee\bar{x}=0.7\vee 0.3=0.7\neq 0.5=0.5\vee 0.5=y\vee\bar{y}$, contradicting to $x\vee\bar{x}=\tilde{1}$ is a constant.

Kalman [2] has proved that the subdirectly irreducibles in **DM** are M_0, M_1 and M_2.

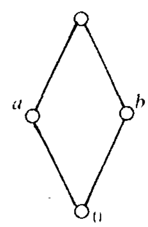
The Hasse diagrams of M_0, M_1 and M_2 are depicted in figure 2, 3, and 4 respectively.

Figure 2.



ith $\tilde{1}=1$

Figure 3.



with $\bar{a}=a$, $\bar{b}=b$

Figure 4.

Theorem 4.1 The subdirectly irreducibles in **M** are M_0 and M_1.

Proof Obviously M_0 and M_1 are the subdirectly irreducibles.

To prove M_2 is not subdirectly irreducible we show that M_2 is not a member of **M**.

Suppose M_2 is a Medium algebra. For $a\in M$, we have $\bar{a}=a$ and $a+\bar{a}+\tilde{a}=1$, thus $a+\tilde{a}=1$, which implies $\tilde{a}=b$. On the other hand, $\tilde{\bar{a}}=a+\bar{\bar{a}}=a$, so $\tilde{1}=\tilde{0}=\tilde{a}\tilde{\bar{a}}=ab=0$. Therefore $0=\tilde{1}=\tilde{0}=\tilde{\bar{1}}=1+\bar{\bar{1}}=1$, a contradiction. Hence M_2 is not a Medium algebra. This completes the proof.

Corollary Let $M\in$ **M** Then M is a subdirect product of copies of M_0 and M_1.

5. Relations with other algebras

Theorem 5.1 Every Medium algebra is a Kleene algebra.

Proof Let M be a Medium algebra, obviously M is a De Morgan algebra. For all $x,y\in M$, $x\bar{x}\leqslant\tilde{x}$ and $x\bar{x}\leqslant x+\bar{x}=\tilde{\bar{x}}$ implies $x\bar{x}\leqslant\tilde{x}\tilde{\bar{x}}=\tilde{0}=\tilde{y}\tilde{\bar{y}}\leqslant\tilde{\bar{y}}=y+\bar{y}$. Therefore M is a Kleene algebra.

Theorem 5.2 Let M be a Medium algebra. Then M is a Boolean algebra if and only if $\tilde{1}=1$.

Proof If $\tilde{1}=1$ then $\tilde{x}\tilde{\bar{x}}=\tilde{1}=1$ implies $\tilde{\bar{x}}=x+\bar{x}=1$ and $x\bar{x}=0$ Thus $\bar{x}$ is the complement of x. Therefore M is a Boolean algebra.

Conversely, if M is a Boolean algebra, then $\tilde{1}=\tilde{\bar{\bar{1}}}=\tilde{1}+\bar{\tilde{1}}=1$.

Definition 5.1 Let L be a De Morgan algebra. $\mathscr{C}(L)=\{x\in L \mid x+\bar{x}=1, x\bar{x}=0\}$ is called the centre of L.

Theorem 5.3 Let M be a Medium algebra. then $\forall x\in M$, $x\in\mathscr{C}(M)$ if and only if $\tilde{x}=\tilde{0}$.

Proof If $x\in\mathscr{C}(M)$, then $\tilde{x}=\tilde{x}+x(x+\bar{x})=\tilde{x}\,\tilde{\bar{x}}+x\tilde{\bar{x}}+\tilde{x}\,\bar{x}=\widetilde{x\bar{x}}=\tilde{0}$.

Conversely, if $\tilde{x}=\tilde{0}$, then $\tilde{\tilde{x}}=\tilde{\tilde{0}}=1$ which implies $x+\bar{x}=1$ and $x\bar{x}=0$, therefore $x\in\mathscr{C}(M)$.

Theorem 5.4 Let M be a Medium algebra such that $\bar{\tilde{0}}=\tilde{0}$. Then $M^{p}=\{x\in M \mid x=a\tilde{0}+b,\ a,b\in\mathscr{C}(M)\}$ is a Post algebra and $\mathscr{C}(M^{p})=\mathscr{C}(M)$.

Proof Straightforward.

Theorem 5.5 Let $M=\langle M, +, \cdot, ^{-}, ^{\sim}, 0, 1\rangle$ be a Medium algebra, $\tilde{M}=\{\tilde{x} \mid x\in M\}$. Then $\langle \tilde{M}, +, \cdot, ^{\sim}, \tilde{0}, 1\rangle$ is a Boolean algebra.

Proof For $x\in M$, $\tilde{0}=\tilde{x}\,\tilde{\tilde{x}}\leqslant\tilde{x}\leqslant 1=\tilde{\tilde{0}}\in M$ implies that $\tilde{0}$ and 1 are the lowest and greatest element in M respectively. For $x, y\in M$, since $\tilde{0}\leqslant\tilde{x}\,\tilde{y}\leqslant\tilde{x}+\tilde{y}$, we have $\widetilde{\tilde{x}\,\tilde{y}}\geqslant\widetilde{\tilde{x}+\tilde{y}}$. Thus, $\tilde{\tilde{x}}+\tilde{\tilde{y}}=\widetilde{\tilde{x}+\tilde{y}}+\widetilde{\tilde{x}\,\tilde{y}}=\widetilde{\tilde{x}\,\tilde{y}}$ and $\tilde{\tilde{x}}\,\tilde{\tilde{y}}=\widetilde{\tilde{x}+\tilde{y}}$ $\widetilde{\tilde{x}\,\tilde{y}}=\widetilde{\tilde{x}+\tilde{y}}$. which implies "$\sim$" satisfies the De Morgan law on $\tilde{M}$. Also we have $\tilde{x}+\tilde{y}=\tilde{\tilde{\tilde{x}}}+\tilde{\tilde{\tilde{y}}}=\widetilde{\tilde{\tilde{x}}\,\tilde{\tilde{y}}}\in\tilde{M}$, $\tilde{x}\,\tilde{y}=\tilde{\tilde{\tilde{x}}}\,\tilde{\tilde{\tilde{x}}}=\widetilde{\tilde{\tilde{x}}+\tilde{\tilde{y}}}\in\tilde{M}$, $\tilde{\tilde{\tilde{x}}}=\tilde{x}$, $\tilde{x}\,\tilde{\tilde{x}}=\tilde{0}$, $\tilde{x}+\tilde{\tilde{x}}=1$. Therefore $\langle M, +, \cdot, ^{\sim}, \tilde{0}, 1\rangle$ is a Boolean algebra.

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中　介　代　数

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摘　要

近年来，朱梧槚和肖奚安在研究中介逻辑演算和中介公理集合论时引进了被排斥在经典逻辑之外的模糊否定词“～”，讨论了反映模糊现象的“中介原则”。因此，中介逻辑是经典的二值逻辑的推广和发展。我们知道，Boole代数是对应于二值逻辑的代数结构．本文提出中介逻辑的代数抽象——中介代数．本文讨论了中介代数组成的等式类(即代数簇)的代数性质，证明了每个中介代数都是 2 和 3 的亚直积．本文还讨论了中介代数的余积，中介代数与Kleen代数、Post代数以及Boole代数之间的关系．

所谓中介代数是一个(2,2,1,1,0,0)型的代数$(M, +, \cdot, {}^{-}, {}^{\sim}, 0, 1)$．其中$(M, +, \cdot, {}^{-}, 0, 1)$组成 De Morgan 代数．而一元运算“～”还满足以下公理：$\forall x, y \in M$，

（1）　$\widetilde{\overline{x}} = \widetilde{x}$.

（2）　$\widetilde{\widetilde{x}} = x + \overline{x}$.

（3）　$x\overline{x} + \widetilde{x} = \widetilde{x}$.

（4）　$\widetilde{x + y} = \widetilde{x}\,\widetilde{y} + \widetilde{x}\,\overline{y} + \overline{x}\,\widetilde{y}$.

（5）　$\widetilde{xy} = \widetilde{x}\,\widetilde{y} + \widetilde{x}y + x\widetilde{y}$.

（6）　$x + \overline{x} + \widetilde{x} = 1$.

从这些公理中可以看出，“中介原则”在中介代数中得到了反映．