Journal of Mathematical Research and Exposition
Vol. 10, No.2, May, 1990

The Djoković's Conjecture on an Integral Inequality*

Hu Guanchu *Tang Jiankang*

(Department of Mathematics, Hangzhou University)

1. Introduction

In 1965, D. Ž. Djoković proposed the following proposition[1]: Let $x_0<x_1<\cdots<x_n$ be real numbers and

$$f(x; x_0, x_1, \cdots, x_n)=(x-x_0)(x-x_1)\cdots(x-x_n), \quad (1)$$

$$M=\max_{x_0<x<x_n} |f(x; x_0, x_1, \cdots, x_n)|, \quad (2)$$

$$\Phi(x_0, x_1, \cdots, x_n)=\frac{1}{M}\int_{x_0}^{x_n} f(x; x_0, x_1, \cdots, x_n)dx,$$

prove (or disprove) the inequality:

$$(-1)^{k+1}(\partial\Phi/\partial x_k)>0. \quad (3)$$

In 1966, The editor of the Amer. Math. Monthly pointed out [2]: From a reexamination of the original proposal there is reason to think that the inequality should have been stated as

$$(-1)^{n+1-k}(\partial\Phi/\partial x_k)>0. \quad (4)$$

For $n=1$ $(k=0,1)$, D.A.Hejhol showed that the inequality (3) is untrue, but (4) is true.

Up to the present, no other solutions or comments have been obtained. D. S. Mitriović and P. M. Vasić called (4) the Djoković's conjecture (see [3], Chapter 3, 7.50).

The purpose of this paper is to disprove the conjecture.

2. Main Result

$f(x; x_0, x_1, \cdots, x_n)$ is called $f(x)$ for short. Let $\bar{x}=g(x_0, x_1, \cdots, x_n)$, $x_l<\bar{x}<x_{l+1}$ be a real zero point of $f'(x)$, such that

$$M=|f(\bar{x})|=\max_{x_0<x<x_n}|f(x)|.$$

We have

$$M=(\bar{x}-x_0)\cdots(\bar{x}-x_l)(x_{l+1}-\bar{x})\cdots(x_n-\bar{x}),$$

thus

* Received Dec. 22, 1987.

$$M^2\partial\Phi/\partial x_k = -M\int_{x_0}^{x_n}\frac{f(x)}{x-x_k}\,dx - M_{x_k}\int_{x_0}^{x_n} f(x)\,dx. \qquad (5)$$

where $M_{x_k} = \partial M/\partial x_k$.

We discuss the following three cases:

(Ⅰ) $n = 2$. for convenience (without less of generality), let $x_0 = 0$, $x_1 = a$, $x_2 = 1$, then $f(x) = x(x-a)(x-1)$. Find the extreme point $\bar{x}$ of $|f(x)|$,

If $0 < a < \frac{1}{2}$, then $\frac{1}{2} < \bar{x} = (1 + a + \sqrt{1 - a + a^2})/3 < 1$, and thus

$$M = \bar{x}(\bar{x} - a)(1 - \bar{x}).$$

We also find that

$$M_{x_0} = -(\bar{x} - a)(1 - \bar{x}),$$
$$M_{x_1} = -\bar{x}(1 - \bar{x}),$$
$$M_{x_2} = \bar{x}(\bar{x} - a).$$

From this it follows that

$$M^2\partial\Phi/\partial x_0 = (\bar{x} - a)(1 - \bar{x})\int_{x_0}^{x_2}(x - \bar{x})(x - a)(x - 1)\,dx$$

where $(\bar{x} - a)(1 - \bar{x}) > 0$,

$$\int_0^1 (x - \bar{x})(x - a)(x - 1)\,dx = [-1 + 2a - 6a^2 + 2(1 - 3a)\sqrt{1 - a + a^2}]/36.$$

Let $-1 + 2a - 6a^2 + 2(1 - 3a)\sqrt{1 - a - a^2} = 0$, we find its unique real root

$$a_1 = [8 + (-109 + \sqrt{109^2 + 512})^{1/3} - (109 + \sqrt{109^2 + 512})^{1/3}]/18$$
$$\doteq 0.1824879.$$

from which it follows that

$$\frac{\partial\Phi}{\partial x_0}\begin{cases} < 0, & \text{if } a > a_1, \\ > 0, & \text{if } a < a_1. \end{cases}$$

Furthermore

$$M^2\partial\Phi/\partial x_1 = \bar{x}(1 - \bar{x})\int_{x_0}^{x_2} x(x - \bar{x})(x - 1)\,dx,$$

where $\bar{x}(1 - \bar{x}) > 0$,

$$\int_0^1 x(x - \bar{x})(x - 1)\,dx = (-1 + 2a + 2\sqrt{1 - a + a^2})/36 > 0, \quad (\text{for all } 0 < a < 1).$$

from this it follows

$$\frac{\partial\Phi}{\partial x_1} > 0, \quad (\text{for all } 0 < a < 1).$$

Furthermore

$$M^2\partial\Phi/\partial x_2 = -\bar{x}(\bar{x} - a)\int_{x_0}^{x_2} x(x - a)(x - \bar{x})\,dx,$$

$$\int_0^1 x(x - a)(x - \bar{x})\,dx = [5 - 10a + 6a^2 + 2(3a - 2)\sqrt{1 - a + a^2}]/36 > 0,$$

We now have that

$$\frac{\partial \Phi}{\partial x_2} < 0.$$

Thus we conclude that the inequality (4) is ture for $a_1 < a < \frac{1}{2}$, and unture for $0 < a < a_1$.

If $a > \frac{1}{2}$, the extreme point $\bar{x}$ of $|f(x)|$ is

$$0 < \bar{x} = (1 + a - \sqrt{1 - a + a^2})/3 < \frac{1}{2}.$$

According to symmetry principle

$$\frac{\partial \Phi}{\partial x_0} < 0,$$

$$\frac{\partial \Phi}{\partial x_1} > 0,$$

$$\frac{\partial \Phi}{\partial x_2} \begin{cases} < 0, & \text{if } a < a_2, \\ > 0, & \text{if } a > a_2. \end{cases}$$

where $a_2 = 1 - a_1 \doteq 0.8175121$.

From this it follows that form (4) is ture for $\frac{1}{2} < a < a_2$, is unture for $a_2 < a < 1$.

Now we have the following theorem

Theorem I Let $x_0 < x_1 < x_2$ be real numbers, f, M, Φ, a_1 and a_2 are defined as above $(n = 2)$, then the inequality (4) is ture for $a_1(x_2 - x_0) < x_1 - x_0 < a_2(x_2 - x_0)$, and it is untrue for $x_1 - x_0 < a_1(x_2 - x_0)$ or $x_1 - x_0 > a_2(x_2 - x_0)$. Similarly, for $n > 2$, the tureness of the inequality (4) depends on the distribution of $x_0 < x_1 < \cdots < x_n$.

(II) Let $x_0 < x_1 < \cdots < x_n$ are $n + 1$ uniformly spaced points of the colsed interval $[x_0, x_n]$. Note $x_k = x_0 + kh$, $(k = 0, 1, \cdots, n)$, with

$$h = (x_n - x_0)/n. \tag{6}$$

Introducing the new variable t such that $x = x_0 + th$, thus $x - x_k = (t - k)h$, and the integral gives

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x = h^{n+2}\int_0^n t(t-1)(t-2)\cdots(t-n)\mathrm{d}t.$$

We write

$$\omega(t) = t(t-1)(t-2)\cdots(t-n)$$

and

$$\varphi(s) = \int_0^s \omega(t)\mathrm{d}t.$$

At frist we discuss n is even $(n = 2m,\ m = 1, 2, \cdots)$.

Lemma I [4] For $n = 2m$, $\varphi(0) = \varphi(n) = 0$.

So (5) is that (observing (6)),

$$M^2 \partial\Phi/\partial x_k = -Mh^{n+1}\int_0^n \frac{t(t-1)\cdots(t-n)}{t-k}\mathrm{d}x.$$

It is well-known as cofficients of Newton-Cotes' integration formulas [4]:

$$A_k^n = \frac{(-1)^{n-k}}{nk!\,(n-k)!}\int_0^n \frac{t(t-1)\cdots(t-n)}{t-k}\mathrm{d}t\ .$$

From this it follows

$$(-1)^{n+1-k} M^2 \partial\Phi/\partial x_k = M h^{n+1} nk!(n-k)!\,A_k^n . \qquad (7)$$

For $n=1,2,\cdots,6$, we find $A_k^n>0$, (see [4], P.242. the table of Newton-Cotes' formulas), so (4) is ture. For large n, some of the values A_k^n become negative, so (4) is untrue.

Now, we discuss that n is odd ($n=2m+1$, $m=1,2,\cdots$), let

$$\bar{x} = x_0 + \bar{t}h$$

From (5) it follows that

$$M^2\partial\Phi/\partial x_k = \pm h^{2n+2}|\bar{t}(\bar{t}-1)\cdots(\bar{t}-k+1)(\bar{t}-k-1)\cdots(\bar{t}-n)|\int_0^n \frac{t(t-1)\cdots(t-n)}{t-k}(t-\bar{t})\mathrm{d}t \qquad (8)$$

It is positive, if $x_k < x_i$, it is negative, if $x_k > x_{k+1}$.

Consider the integration

$$I = \int_0^n \frac{t(t-1)\cdots(t-n)}{t-k}(t-\bar{t})\,\mathrm{d}t$$

and note $t(t-1)\cdots(t-n) = \dfrac{\Gamma(t+1)}{\Gamma(t-n)}$, where $\Gamma(t)$ is Gamma function.

Apply remainder element formula of $\Gamma(t)$ [5]:

$$\Gamma(t)\Gamma(1-t) = \pi/\sin(\pi t)$$

thus

$$\frac{1}{\Gamma(t-n)} = \frac{\Gamma(n+1-t)\sin(\pi(t-n))}{\pi} = (-1)^n\frac{\Gamma(n+1-t)\sin\pi t}{\pi}$$

from which it follows that

$$I = (-1)^n\int_0^n \frac{\Gamma(t+1)\Gamma(n+1-t)\sin\pi t}{\pi(t-k)}(t-\bar{t})\,\mathrm{d}t$$

We express the integration as sum of the three parts

$$\int_0^n = \int_0^3 + \int_3^{n-3} + \int_{n-3}^n = A + B + C .$$

At first we discuss B. The derivate of logarithm of $\Gamma(t)$ is a increasing function with t, the

$$\frac{\Gamma'(t+1)}{\Gamma(t+1)} - \frac{\Gamma'(n+1-t)}{\Gamma(n+1-t)} \begin{cases} <0, & \text{for} \quad -1<t<n/2, \\ >0, & \text{for} \quad n/2<t<n+1. \end{cases}$$

From this, we obtain ln $\Gamma(t+1)\Gamma(n+1-t)$, then $\Gamma(t+1)\Gamma(n+1-t)$ attain to largest in the end of the interval $[3, n-3]$, thus $0<\Gamma(t+1)\Gamma(n+1-t)<\Gamma(4)\Gamma(n-2) = 6\Gamma(n-2)$,

$$\left|\frac{\sin\pi t}{\pi(t-k)}\right| = \left|\frac{\sin\pi(t-k)}{\pi(t-k)}\right| < 1, \text{ for all } 0<t<n,$$

and $|t-\bar{t}|<n$.

Finally, we get a bound for absolute value of B,

$$|B| < 6n^2\Gamma(n-2) = \frac{6n^2\Gamma(n+1)}{n(n-1)(n-2)} = O\left(\frac{\Gamma(n+1)}{n}\right).$$

Now we discuss A. By expressing $\ln\Gamma(n+1-t)$ as Taylor's series about $n+1$.

$$\ln\Gamma(n+1-t) = \ln\Gamma(n+1) - t\frac{\Gamma'(n+1)}{\Gamma(n+1)} + O\left(\frac{1}{n}\right), \qquad (9)$$

for sufficiently large n, we have that [5],

$$\frac{\Gamma'(n+1)}{\Gamma(n+1)} = \psi(n+1) = \ln(n+1) + O\left(\frac{1}{n^2}\right),$$

thus

$$\Gamma(n+1-t) = \Gamma(n+1)e^{-t\ln(n+1)}\left[1+O\left(\frac{1}{n}\right)\right],$$

for $0<t<3$, then $0<\Gamma(t+1)<6$, and $\sin\pi t = \pi t - (\pi t)^3/3 + (\pi t)^5/5 - \cdots$,

$$\frac{1}{\pi(t-k)} = -\frac{1}{\pi k(1-t/k)} = -\frac{1}{k\pi}(1+t/k+t^2/k^2+\cdots),$$

from this it follows that $(1<k<n-1)$

$$\Gamma(t+1)\frac{\sin\pi t}{(t-k)\pi} = -\frac{6t}{k} + O\left(\frac{t^2}{k^2}\right),$$

thus

$$A = \int_0^3 \Gamma(n+1)e^{-t\ln(n+1)}\left[1+O\left(\frac{1}{n}\right)\right]\left[-\frac{6t}{k}+O\left(\frac{t^2}{k^2}\right)\right](t-\bar{t})\,dt,$$

since

$$\int_0^3 e^{-t\ln(n+1)}t\,dt = \frac{1}{\ln^2(n+1)} - \frac{1}{(n+1)^3}\left[\frac{3}{\ln(n+1)} + \frac{1}{\ln^2(n+1)}\right],$$

$$\int_0^3 e^{-t\ln(n+1)}t^2\,dt = \frac{2}{\ln^3(n+1)} - \frac{1}{(n+1)^3}\left[\frac{9}{\ln(n+1)} + \frac{1}{\ln^2(n+1)} + \frac{2}{\ln^3(n+1)}\right].$$

from this it follows that

$$A = \frac{6\Gamma(n+1)}{k\ln^2(n+1)}\left[\bar{t} + O\left(\frac{1}{\ln(n+1)}\right)\right],$$

At last, we discuss C. Let $t=n-s$, $\bar{t}=n-\bar{s}$, hence

$$C = \int_{n-3}^{n} \frac{\Gamma(t+1)\Gamma(n+1-t)\sin\pi t}{\pi(t-k)}(t-\bar{t})\,dt$$

$$= \int_3^0 \frac{\Gamma(n+1-s)\Gamma(s+1)\sin\pi(n-s)}{\pi(n-k-s)}(\bar{s}-s)\,d(-s)$$

$$= (-1)^{n+1}\int_0^3 \frac{\Gamma(n+1-s)\Gamma(s+1)\sin\pi s}{\pi(s-(n-k))}(s-\bar{s})\,ds.$$

Similar to A, we obtain

$$C=(-1)^{n+1}\frac{6\Gamma(n+1)}{(n-k)\ln^2(n+1)}[n-\bar{t}+O(\frac{1}{\ln(n+1)})].$$

Since

$$\lim_{n\to\infty}\frac{\ln^2(n+1)}{n}=0.$$

thus according to the absolute value, the B is smaller than A and C. Finally, we get an estimate of I.

$$I=(-1)^n\frac{6\Gamma(n+1)}{\ln^2(n+1)}[\frac{\bar{t}}{k}+(-1)^{n+1}\frac{n-\bar{t}}{n-k}][1+O(\frac{1}{\ln(n+1)})],\tag{10}$$

if $n=2m+1$ is odd, then

$$I=(-1)\frac{6\Gamma(n+1)}{\ln^2(n+1)}[\frac{\bar{t}}{k}+\frac{n-\bar{t}}{n-k}][1+O(\frac{1}{\ln(n+1)})]<0.$$

From this and (8), we can deduce the result, if $x_i<\bar{x}<x_{i+1}$, then

$$M^2\partial\Phi/\partial x_k\begin{cases}<0, & \text{for}\quad 0<k\leqslant i,\\ >0, & \text{for}\quad i+1<k\leqslant n.\end{cases}$$

thus (4) is unture.

We now conclude the following theorem.

Theorem 2 Let $x_0<x_1<\cdots<x_n$ are $n+1$ uniformly spaced points of the closed interval $[x_0,x_n]$, then for $n\leqslant 6$, form (4) is ture, for large n, (4) is untrue.

(Ⅲ) We broaden the scope of our examination by considering integral of the form

$$\Phi(x_0,x_1,\cdots,x_n)=\frac{1}{M}\int_a^b\rho(x)f(x)\mathrm{d}x,\tag{11}$$

where $\rho(x)$ is a given nonnegative weight function on the interval $[a,b]$.

Let $x_0<x_1<\cdots<x_n$ be the roots of $(n+1)th$ orthogonal polynomial $P_{n+1}(x)$.

$$\int_a^b\rho(x)P_{n+1}(x)P_k(x)\mathrm{d}x=0,\ \text{if}\quad k\leqslant n.$$

where $P_k(x)$ is a k degree polynomial, all the roots x_i $(i=0,1,\cdots,n)$ lie in the open interval (a,b), $f(x)$ and M is defined as above. Now in the right of (5) the second item is zero, thus

$$M^2\partial\Phi/\partial x_k=-M\int_a^b\rho(x)\frac{f(x)}{x-x_k}\mathrm{d}x.$$

It is the well-known cofficients of the Gaussian integration rules [4]:

$$C_k^n=\int_a^b\rho(x)\frac{f(x)}{(x-x_k)f'(x_k)}\mathrm{d}x=\int_a^b\rho(x)[\frac{f(x)}{(x-x_k)f'(x_k)}]^2\mathrm{d}x>0$$

where

$$f'(x_k)=(-1)^{n-k}(x_k-x_0)\cdots(x_k-x_{k-1})(x_{k+1}-x_k)\cdots(x_n-x_k),$$

from this it follows that

$$(-1)^{n+1-k}\partial\Phi/\partial x_k=(-1)^{2(n+1-k)}\frac{1}{M}(x_k-x_0)\cdots(x_k-x_{k-1})(x_{k+1}-x_k)\cdots(x_n-x_k)C_k^n>0.$$

The most common weight function $\rho(x)\equiv 1$, and the interval $[-1,1]$, the corresponding orthogonal polynomials are the Lagendre polynomials:

$$P_k(x)=\frac{k!}{(2k)!}\frac{d^k}{dk^k}(x^2-1),\quad k=0,1,\cdots,n+1,\cdots$$

Now we arrive at the main result of this section,

Theorem 3 Let $a<x_0<x_1<\cdots<x_n<b$ be the roots of the $(n+1)$-*th* orthogonal polynomial $P_{n+1}(x)$, associated with the weight function $\rho(x)$ in the interval $[a, b]$, where

$$\Phi(x_0,x_1,\cdots,x_n)=\frac{1}{M}\int_a^b\rho(x)f(x)dx,$$

then (4) is true.

3. Summary

We have shown that for any distribution of $x_0<x_1<\cdots<x_n$ $(n\geq 2)$, the Djoković's conjecture is untrue. It is true only for some special cases, as shown in the Theorem 1, 2, 3.

References

[1] Djoković, D.Z., Problem 5311, Amer. Math. Monthly, 72:7(1965), 794.

[2] Djoković, D.Z., Amer. Math. Monthly, 73:7(1966), 788.

[3] Mitrinovic, D.S.& Vasić, P.M., Analytic Inequalities, Springer-Verlag, 1970. (中译本:解析不等式，科学出版社, 1986).

[4] Березин Н.С.И Жидков Н.П. Методы Выцислений 1,Физматтиз. Москва, 1959.

[6] Erdelyi, A.ed., Higher Transcendental Function, McGraw-Hill, 1953.

关于一个积分不等式的Djoković猜想

胡冠初　　汤健康

（杭州大学数学系）

摘　　要

在1965年，Djoković，D. Z 提出[1]：设 $x_0<x_1<\cdots<x_n$ 是 $n+1$ 个实数，且

$$f(x;x_0,x_1,\cdots,x_n)=(x-x_0)(x-x_1)\cdots(x-x_n),\qquad (1)$$

$$M=\max_{x_0<x<x_n}|f(x;x_0,x_1,\cdots,x_n)|\qquad (2)$$

$$\Phi(x_0,x_1,\cdots,x_n)=\frac{1}{M}\int_{x_0}^{x_n}f(x;x_0,x_1,\cdots,x_n)\mathrm{d}x,\qquad (3)$$

则有不等式：

$$(-1)^k\partial\Phi/\partial x_k>0.$$

[2]中指出上式应改为

$$(-1)^{n+1-k}\partial\Phi/\partial x_k>0.\qquad (4)$$

这个不等式至今没有证明，以至Mitrinović等，将它作为猜想编入《解析不等式》一书中[3]，

本文的目的是否定这个猜想，我们有：

定理1　对 $n=2$，$x_0<x_1<x_2$，当 $a_1(x_2-x_0)<x_1-x_0<(1-a_1)(x_2-x_0)$ ($a_1=0.1824879\cdots$)时，(4)式成立，当 $x_1-x_0<a_1(x_2-x_0)$ 或 $x_1-x_0>(1-a_1)(x_2-x_0)$ 时，(4)不成立，一般说，(4)式是否成立和点 $x_0<x_1<\cdots<x_n$ 的分布情况有关．

定理2　设 $x_0<x_1<\cdots<x_n$ 是 $n+1$ 个等距分布的点．则当 $n<6$ 时，(4)式成立，对较大的 n，(4)不成立．

定理3　将(3)式改为

$$\Phi(x_0,x_1,\cdots,x_n)=\frac{1}{M}\int_a^b\rho(x)f(x;x_0,x_1,\cdots,x_n)\mathrm{d}x$$

其中 $\rho(x)$ 为非负权函数，设 $a<x_0<x_1<\cdots<x_n<b$ 是 $n+1$ 次正交多项式（具有权 $\rho(x)$）的零点．则相应的(4)式成立．

总之，对任意分布的 $x_0<x_1<\cdots<x_n$，(4)式是不成立的，只对一些特殊情形，(4)式成立．如定理1，2，3所述．