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Differential Stability for Parametric Nonsmooth Optimization*

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Abstract

This paper studies the continuous and differential properties of marginal functions or extremal value functions of nonsmooth optimization problems with vecter parameter. It gives the bounds of some directional derivatives of marginal functions for nonsmooth and nonconvex problems in which the target function and inequality constraints are Lipschitz and equality constraints are smooth. It is an extension of [1], and [4] from one-side perturbation to vecter parametric perturbation. It also considers an extension of [1] from smooth functions to nonsmooth functions.

Section I Introduction

We consider the parametric nonsmooth optimization problem:

Pt: $\min f(x)$

$$\text{s.t.}\quad x\in C;$$
$$g_i(x,t)\leqslant 0,\qquad i=1,\cdots,p;$$
$$h_j(x,t)=0,\qquad j=p+1,\cdots,q.$$

where $f:R^n\to R$, $g_i:R^n\times R^m\to R$, $i=1,\cdots,p$, are Lipschitz functions. $h_j:R^n\times R^m\to R$, $j=p+1,\cdots,q$, are continuous differentable functions. C is a convex compact set.

Remark The hypothesis on the boundedness of the set C makes convenient for us to consider the continuity of optimal value function, and it is sufficient for the differential stability of optimal value function to consider that C is a closed convex set. Moreover, it is natural that we may assume that h_j are smooth functions, because it is rare for algorithm to have nonsmooth conditions.

The vecter $t=(t_1,\cdots,t_m)$ of programming (Pt) is parametric vecter of the problem. The feasible set $S:R^m\to R^n$ of the programming (Pt).

$$S(t)\triangleq\{x\in C\,|\,g_i(x,t)\leqslant 0,\ h_j(x,t)=0,\ i=1,\cdots p,\ j=p+1,\cdots,q\}\qquad(1)$$

For each parameter t, the optimal value $V:R^m\to R$ for the programming (Pt) is:

$$V(t)\triangleq\min\{f(x)\,|\,x\in S(t)\}.\qquad(2)$$

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The corresponding optimal solution set $M: R^m \to R^n$

$$M(t) \triangleq \{\bar{x} \in S(t) \mid f(x) = V(t)\} \tag{3}$$

What we study is the continuity of the function $V(t)$ and set-value map $M(t)$ and the properties of some directional derivatives about $V(t)$.

In the case of all functions of the programming being convex, Rockafellar [13] had quite well studied the stability and differential stability of the programming in 70's. Zencke and Hettich [12] in 1987 had made a thorough discussion to the programming in which all functions are convex and smooth. Evans [7] firstgave the stable analysis for optimization problems whose functions are non-convex but smooth or have more properties. Ref. [8,9,10,11] have respectively made a progressive discussion for infinite dimension programme or second-order directional derivative of the optimal Value function. We are interested in the optimization problem which functions are nonconvex and nonsmooth. Rockafellar [3] has given a sophisticated study for the optimization which has vecter parameter and singleside perturbation using generalized lagrange function. Ref. [1], [4] have given the upper and lower bounds of some directional derivatives of marginal function with two different methods. What they thought of is an easy problem for it has only single side perturbation. It can be seen from above that the studies for nonsmooth optimization are not much.

Section 2 Stability

In the following, F will represent a point-to-set mapping of R^m into R^n.

The map F is upper semi-continuous at $t_0 \in R^m$, if for any neighbourhood $N(F(t_0))$ containing $F(t_0)$, there is a neighbourhood $N(t_0)$ of t_0 such that:

$$F(N(t_0)) \subset N(F(t_0)).$$

The map F is said to be closed at $t_0 \in R^m$, if for any $t_n \to t_0$, $x_n \in$ t_n), and $x_n \to x_0$, we have $x_0 \in F(t_0)$.

The map F is uniform compact near t_0 if there is a neighborhood $N(t_0)$ such that the set Co $\{\bigcup_{t \in N(t_0)} F(t)\}$ is campact.

Property 2.1 [2]. Let F be uniformly compact near t_0, then F is closed at t_0 if and only if $F(t_0)$ is compact and F is upper semicontinuous at t_0.

It is obvious that the map $S: R^m \to R^n$ defined by (1) is closed and upper semicontinous.

Definition 2.1 The optimization problem (Pt) is called to be stable at t_0 if (Pt) satisfies the following three conditions at t_0:

(1) $V(t)$ is continuous at t_0;

(2) The map M is closed at t_0;

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(3) The set $M(t)$ is nonempty for any $t\in N(t_0)$, $N(t_0)$ is a neighbourhood of t_0.

Lemma 2.1 If the feasible set $S(t)$ is nonempty for some parametric vecter t, then the optimal solution set $M(t)$ is also nonempty.

Proof: Obvious.

Lemma 2.2 Considering the programme (Pt). If $S(t)$ is nonempty, then:

(1) $V(t)$ defined by formula (2) is continuous;

(2) $M(t)$ defined by formula (3) is closed and upper semi-continuous.

Proof (1) Taking $x_0\in M(t_0)$, then $f(x_0)=V(t_0)$, for any $\varepsilon>0$ because $f(x)$ is continuous, there is a neighborhood $N(x_0)$ of x_0 suce that:

$$f(x_0)-\varepsilon\leqslant f(x)\leqslant f(x_0)+\varepsilon, \quad \text{for any } x\in N(x_0)$$

For $N(x_0)$, because $S(t_0)$ is upper semi-continuous, there is a neighborhood $N(t_0)$ of t_0, such that:

$$S(N(t_0))\subset N(x_0).$$

Hence, $V(t_0)-\varepsilon<V(t_0)<V(t_0)+\varepsilon$, fof any $t\in N(t_0)$.

Therefore, $V(t)$ is continuous at t_0.

(2) The result is obtained by combining (1) and the closeness of $S(t)$.

Theorem 2.1 If the programme (Pt) is feasible, then it is stable.

Remark In Ref. [2], Gauvin studied the smooth programme, the target function of which also has parameter, its stability requires (M—F) regularity assumption: There is a $\bar{r}\in R^n$ such that the following two conditions are satisfied:

(1) $\langle \nabla_x g_i(\bar{x},\bar{t}),\bar{r}\rangle<0,\ \ i\in I(\bar{x},\bar{t})=\{i\,|\,g_i(\bar{x},\bar{t})=0\}$

$\langle \nabla_x h_j(\bar{x},\bar{t}),\bar{r}\rangle=0,\ \ j=p+1,\cdots,q.$

(2) The Jacabian matrix $[\nabla_x h_j(\bar{x},\bar{t})]$ has line rank $(q-p)$.

But for the nonsmooth programme which we are considering, the assumption is not required.

Section 3 Optimal condition

Making an equivalent change to (Pt), we obtain:

Pt′ $\min f(x)$

$$\begin{aligned} \text{s.t.}\quad & (x,y)\in C\times R^m \\ & g_j(x,y)\leqslant 0, \quad i=1,\cdots,p \\ & h_j(x,y)=0. \quad j=p+1,\cdots,q \\ & -y_l+t_l=0 \quad l=1,\cdots,m \end{aligned}$$

It is easy to see that the feasible set, optimal solusion set and optimal value function of the new programme (Pt′) are the same as the programme (Pt)′s. So we still describe them using original signs. They are maps responding to t. The new programme (Pt′) is also a parametric programming responding to t.

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For a locally Lipschitz function $f: R^m \to R$, $\partial f(x)$, $f^0(x; h)$ denotes respectively the generalized gradient and generalized directional derivative at x in the sense of Clarke.

The distance function from a point to set C:

$$d_C(\bar{x}) \triangleq \inf\{\|\bar{x} - c\| \mid c \in C\}$$

The Clarke tangent cone of set C at $\bar{x}$:

$$T_C(\bar{x}) = \{v \mid d_C^0(\bar{x}; v) = 0\}$$

The normal cone of set C:

$$N_C(\bar{x}) = \{\xi \mid \langle \xi, v\rangle < 0, \text{ for any } v \in T_C(\bar{x})\}.$$

Hence, $T_C^0(\bar{x}) = N_C(\bar{x})$, where $T_C^0(\bar{x})$ is the polar cone of $T_C(\bar{x})$.

The recession cone of set C:

$$0^+C = \{x + \lambda y \in C, \quad \text{for any } x \in C, \ \lambda > 0\}$$

The tangent compatible cone of set C.

$K_C(\bar{x}) = \{v \mid \text{for any } \varepsilon > 0, \text{ there is a } t \in (0, \varepsilon), w \in v + \varepsilon \cdot B, \text{sunh that } x + tw \in C\}$

Obviously, $T_C(\bar{x}) \supseteq K_C(\bar{x})$.

If above sets are equal, then lset C is called a regular set. All above definitions are quoted from [13], [14].

In the following of this paper, we always assume that set C is a regular set and the optimal solution set of the programming (Pt) at t_0 is nonempty and the optimal value $V(t_0)$ is finite.

Lemma 3 l Given an $\bar{x} \in M(t_0)$, then there exists a nonzero vecter $(\lambda_0, \cdots, \lambda_q, \cdots, \lambda_{q+m})$ such that:

(1) all $\lambda_0, \cdots, \lambda_p$ are not less than zero;

(2) $\lambda_i g_i(\bar{x}, \bar{y}) = 0$, $i = 1, \cdots, p$;

(3) $0 \in \lambda_0 \partial f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \lambda \partial g_i(\bar{x}, \bar{y}) + \sum_{j=p+1}^{p} \lambda_j \cdot \nabla h_j(\bar{x}, \bar{y}) - \sum_{l=g+1}^{g+m} \lambda_l \cdot e_l + N_{C \times R^m}(\bar{x}, \bar{y})$.

Where $\partial f(\bar{x}, \bar{y})$ is the generalized gradient in the sense of Clarke at $(\bar{x}, \bar{y})$, e_l is an $n+m$ dimension vecter. Its elements are zero apart from the $(n+1)th$ element. $\bar{y} = t_0$.

Proof It can be found in Ref. [15]

Lemma 3.2 [14] (1) $N_C(x) = \mathrm{cl}\{\bigcup_{\lambda > 0} \lambda \cdot \partial d_C(x)\}$;

(2) $T_{CXR^m}(x, y) = T_C(x) X T_{R^m}(y)$;

(3) $N_{CXR^m}(x, y) = N_C(x) X N_{R^m}(y)$. ■

Lemma 3.3 [16] If the set C is regular, then $\bigcup_{\lambda > 0} \lambda \cdot \partial d(x)$ is closed. ■

Combining the above two lemmas, we have

$$N_C(\bar{x}) = \bigcup_{\lambda > 0} \lambda \cdot \partial d_C(\bar{x}).$$

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Lemma 3.4 [14] Let $f: S \to R$ be a locally Lipschitz function and its Lipscl constant be K, set $C \subset S$. If the minimum of function f on C is obtained then the minimum of the function $f(x) + K d_C(x)$ on S is also obtained at x. ■

Lemma 3.5 (1) $d_{T_C(\bar{x})}(v) = d^0_{T_C(\bar{x})}(0; v)$, for any v;

(2) $N_{T_C(\bar{x})}(0) = N_C(\bar{x}) = T^0_C(\bar{x})$.

Proof (1) Since that $T_C(x)$ is a convex set and $d_C(x)$ is a convex function, $d_C(x)$ is a regular function. Therefore,

$$d_{T_C(\bar{x})}(0; v) = \lim_{\lambda \to 0} \frac{d_{T_C(\bar{x})}(\lambda \cdot v) - d_{T_C(\bar{x})}(0)}{\lambda}$$

Since $T_C(x)$ contains original point and $T_C(x)$ is a convex cone, then:

$$\begin{aligned} d^0_{T_C(\bar{x})}(0; v) &= \lim_{\lambda \to 0} [d_{T_C(\bar{x})}(\lambda \cdot v)]/\lambda \\ &= \lim_{\lambda \downarrow 0} [\inf\{\|v - t\| \mid t \in T_C(\bar{x})\}]/\lambda \\ &= \inf\{\|v - t\| \mid t \in T_C(\bar{x})\} = d_{T_C(\bar{x})}(v). \end{aligned}$$

(2)

$$\begin{aligned} N_{T_C(\bar{x})}(0) &= \{y \mid \langle y, v\rangle \leq 0, \text{ for any } v \in T_{T_C(\bar{x})}(0)\} \\ &= \{y \mid \langle y, v\rangle \leq 0,\ d^0_{T_C(\bar{x})}(0; v) = 0,\ \forall v\} \\ &= \{y \mid \langle y, v\rangle \leq 0,\ d_{T_C(\bar{x})}(v) = 0, \forall v\} \\ &= \{y \mid \langle y, v\rangle \leq 0, \text{ for any } v \in T_C(\bar{x})] = N_C(\bar{x}) = T^0_C(\bar{x}). \end{aligned}$$ ■

Theorem 3.1 Let $\bar{x} \in M(t_0)$.

$$\Lambda \triangleq \{(\lambda_0, \cdots, \lambda_p, \cdots, \lambda_q, \cdots, \lambda_{q+m}) \mid \lambda_i \geq 0,\ 0 \leq i \leq p;\ \lambda_i g_i(\bar{x}, \bar{y}) = 0,\ 1 \leq i \leq p;$$

$$0 \in \lambda_0 \partial f(\bar{x}, \bar{y}) + \sum_{i=1}^{p} \lambda_i \cdot \partial g_i(\bar{x}, \bar{y}) + \sum_{j=p+1}^{q} \lambda_j \nabla h_j(\bar{x}, \bar{y}) - \sum_{l=q+1}^{q+m} \lambda_l \cdot e_l + N_{CXR^m}(\bar{x}, \bar{y})\}.$$

$$I(\bar{x}, \bar{y}) \triangleq \{i \mid g_i(\bar{x}, \bar{y}) = 0,\ 1 \leq i \leq p\}.$$

$$K \triangleq \bigcup_{T_{CXR^m}(\bar{x}, \bar{y})} \{(k_0, \cdots, k_q, \cdots, k_{q+m}) \mid v \in T_{CXR^m}(\bar{x}, \bar{y}) \text{ satisfies}:$$

$$\begin{aligned} &k_0 \geq f^0(\bar{x}, \bar{y}; v); \\ &k_i \geq g_i^0(\bar{x}, \bar{y}; v), && i \in I(\bar{x}, \bar{y}); \\ &k_j = \langle \nabla h_j(\bar{x}, \bar{y}),\ v\rangle, && p+1 \leq j \leq q; \\ &k_l = \quad v_{l-q+m} = -\langle e_l, v\rangle, && q+1 \leq l \leq q+m\}. \end{aligned}$$

Then: $\Lambda = \{(\lambda_0, \lambda) \mid (\lambda_0, \lambda) K \geq 0\}$ and it has nonzero element.

Proof For any $(\lambda_0, \lambda) \in \{(\lambda_0, \lambda) K \geq 0\}$, the following information tained from the construction of K:

$\lambda_i \geq 0,\ 0 \leq i \leq p$; and for any $0 \leq i \leq p$ if $g_i(\bar{x}, \bar{y}) \leq 0$, then $\lambda_i = 0$. Namely λ_i, $g_i(x, y) = 0, 0 \leq i \leq p$. It can also be seen from the construction of K:

$$F(v) = \lambda_0 f^0(\bar{x}, \bar{y}; v) + \sum \lambda_i S_i^0(\bar{x}, \bar{y}; v) + \sum_{j=p+1}^{q} \lambda_j \cdot \langle \nabla h_j(\bar{x}, \bar{y}), v\rangle - \sum_{l=q+1}^{q+m} \lambda_l \cdot v_{l-q+m} \geq 0,$$

for any $v \in T_{CXR^m}(x, y)$.

Let $\rho > 0$ be a Lipschitz constant of the function $F(v)$ on $R^n \times R^m$, hence by

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lemma 3.4 and lemma 3.5. (1):

$$F(v)+\rho\cdot d_{T_{CXR^m}}(v)\geq 0, \quad \text{for any } v\in R^n\times R^m.$$

$$F(v)+\rho\cdot d_{T_{CXR^m}(\overline{x},\overline{y})}(0;v)\geq 0, \text{ for any } v\in R^n\times R^m.$$

Hence by the seperation theorem:

$$0\in\lambda_0\cdot\partial f(\overline{x},\overline{y})+\sum_{i=1}^{p}\lambda_i\partial g_i(\overline{x},\overline{y})+\sum_{j=p+1}^{q}\lambda_j\nabla h_j(\overline{x},\overline{y})-\sum_{l=q+1}^{q+m}\lambda_l\cdot e_l+\rho\cdot\partial d_{T_{CXR^m}}(0).$$

Since $N_{T_{CXR^m}}(0)\supset\bigcup_{\lambda\geq 0}\lambda\cdot\partial d_{T_{CXR^m}(\overline{x},\overline{y})}(0)$ and by Lemma 3.5. (2):

$$0\in\lambda_0\cdot\partial f(\overline{x},\overline{y})+\sum_{i=1}^{p}\lambda_i\cdot\partial g_i(\overline{x},\overline{y})+\sum_{j=p+1}^{q}\lambda_j\nabla h_j(\overline{x},\overline{y})-\sum_{l=q+1}^{q+m}\lambda_l\cdot e_l+N_{CXR^m}(\overline{x},\overline{y}).$$

Therefore $(\lambda_0,\lambda)\in\wedge$.

For any $(\lambda_0,\lambda)\in\wedge$, then $\lambda_i\geq 0$, $0\leq i\leq p$, and there is a $\xi_0\in\partial f(\overline{x},\overline{y})$; $\xi_i\in\partial g_i(x,y)$, $0\leq i\leq p$; $\overline{\xi}\in\partial d_{CXR^m}(\overline{x},\overline{y})$ and $\rho>0$.

Such that:

$$\lambda_0\xi_0+\sum_{i=1}^{p}\lambda_i\xi_i+\sum_{j=p+1}^{p}\lambda_j\cdot\nabla h_j(\overline{x},\overline{y})-\sum_{l=q+1}^{q+m}\lambda_l\cdot e_l+\rho\cdot\overline{\xi}=0.$$

Multiplying the both sides of the last equality by $V\in T_{CXR^m}(\overline{x},\overline{y})$, and then using the difinition of $T_{CXR^m}(\overline{x},\overline{y})$ and the fact that the Clarke's generalized directi onal derivative is the support function of the generalized gradient, we have

$$\lambda_0 f^0(\overline{x},\overline{y};v)+\sum_{i=1}^{p}\lambda_i g_i^0(\overline{x},\overline{y};v)+\sum_{j=p+1}^{q}\lambda_j\langle\nabla h_j(\overline{x},\overline{y}),v\rangle-\sum_{l=q+1}^{q+m}\lambda_l\cdot\langle e_l,v\rangle\geq 0,$$

for any $v\in T_{CXR^m}(\overline{x},\overline{y})$.

Since $\lambda_i=0$, when $i\in\{1,\cdots,p\}-I(\overline{x},\overline{y})$.

Hence, $(\lambda_0,\lambda)K\geq 0$.

Therefore: $\wedge=\{(\lambda_0,\lambda)K\geq 0\}$.

That the set $\{(\lambda_0,\lambda)\mid(\lambda_0,\lambda)K\geq 0\}$ has nonzero element can be obtained by lemma 3.1. ■

For any $(\lambda_0,\lambda)\in\wedge$, we separate the set $\wedge$ into two sets relying on whether λ_0 is zero.

$$\wedge_0=\{\lambda\mid(0,\lambda)\in\wedge\}$$

$$\wedge_1=\{\lambda\mid(1,\lambda)\in\wedge\}$$

Obviously, $\wedge$ and $\wedge_0$ are closed convex cone, $\wedge_1$ is a closed convex set, and $\wedge_1=\wedge_1+\wedge_0$; $\wedge_0=0^+\wedge_1$, if $\wedge_1\neq\phi$.

Section 4 Differential stability

$L\triangleq\{(t_0,\cdots,t_{q+m})\mid t_i\geq 0,\ 0\leq i\leq p;\ t_j=0,\ p+1\leq j\leq q+m\}$

$t=(t_1,\cdots,t_m)$, $T=(0,\cdots,0,t)\in R^{q+m}$ is the parametric of (Pt')

$L(S,T)\triangleq L+\{a(S,-T)\mid a>0\}$

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$$S(T) \triangleq \sup\{\lambda T | \lambda \in \wedge_1\}$$

$S(T)$ is the support function of $\wedge_1$.

Lemma 4.1 [4] (i) Let T satisfy $\lambda T < 0$, for all nonzero $\lambda \in \wedge_0$, then $S(T) < \infty$ and for any $S > S(T)$, K and $L(S, T)$ are not separable.

(ii) Let T satisfy $\lambda T \geqslant 0$ for all $\lambda \in \wedge_0$ and $S(T) < \infty$, then for any $S > S(T)$, we have $(S, -T) \in \mathrm{cl}(K - L(S, T))$. ■

Now we introduce some directional derivatives and generalized gradients.

The upper Dini derivative:

$$V^+(t_0; t) \triangleq \limsup_{\tau \downarrow 0} [V(\tau t) - V(t_0)]/t$$

The lower directional Hadamard contingent derivative:

$$\mathrm{d}V(t_0; t) \triangleq \lim_{\tau \downarrow 0}, \inf_{t' \to t} [V(\tau t') - V(t_0)]/\tau$$

The generalized directional derivative:

$$v^{\uparrow}(t_0; t) \triangleq \lim_{\tau \downarrow 0} \sup_{\eta_v \to 0} \inf_{\|t' - t\| < \varepsilon} [v(t_0 + \tau t' + \eta) - v(t_0 + \eta)]/\tau$$

Where $\eta_v \to 0$ iff $\eta \to 0$ and $v(\eta) \to v(0)$.

The lower subdifferential:

$$\partial v(t_0) \triangleq \{y | \langle y, t\rangle \leqslant \mathrm{d}v(t_0; t), \quad \forall t\}$$

The generalized gradient:

$$\partial V(t_0) \triangleq \{y | \langle y, t\rangle \leqslant V^{4}(t_0; t), \forall t\}$$

Let $H = (h_{p+1}, \cdots, h_q)$, if the line rank of $\nabla_x H(\overline{x}, \overline{y})$ is $(q-p)$, then the programming (Pt′) is called to be satisfying A-constraint qualification.

Remark The constraint qualification we give makes weaker a requirement to the equal constraints than the $(M-F)$ constraint qualification did and no requirement to inequality constraints.

Lemma 4.2 [2]. Let programming (Pt) satisfy A-constraint qualification, Then for any direction $(v_1, v_2) \in R^n \times R^m$, there is a neighorhood $\mathrm{N}(\overline{x}, \overline{y})$ of $(\overline{x}, \overline{y})$, a neighborhood $N(v_1, v_2)$ of (v, v) and $\tau_0 > 0$ and there is a continuous function: $a(\tau; x, y; v_1, v_2)$ on $\Omega = [-\tau, \tau] \times \mathrm{N}(\overline{x}, \overline{y}) \times \mathrm{N}(v_1, v_2) \to R$ such that: $a(0; x, y; v_1, v_2) = 0$, $(x, y; v_1, v_2) \in \mathrm{N}(x, y) \times \mathrm{N}(v_1, v_2)$;

$$\frac{\mathrm{d}a}{\mathrm{d}\tau}(\tau; x, y; v_1, v_2) \text{ is existent and continuous on } \Omega;$$

$$\frac{\mathrm{d}a}{\mathrm{d}\tau}(0; x, y; v_1, v_2) = 0, \quad (x, y; v_1, v_2) \in \mathrm{N}(\overline{x}, \overline{y}) \times \mathrm{N}(v_1, v_2);$$

and function $X(\tau; x, y; v_1, v_2) = X + \tau v_1 + a(\tau; x, y; v_1, v_2)$ satisfies:

$$\mathrm{H}(X(\tau; x, y; v_1, v_2), y + \tau v_2) = \mathrm{H}(x, y) + \tau\{\nabla_x \mathrm{H}(x, y) v_1 + \nabla_y \mathrm{H}(x, y) v_2\},$$

for any $(\tau; x, y; v_1, v_2) \in \Omega$.

Theorem 4.1 Let $\overline{x} \in M(t_0) \cap \mathrm{int}\, C$.

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(1) If A-constraint qualification is satisfied, and $\lambda T<0$ for any $\lambda\in\wedge_0(\overline{x})$, where $T=(0,\cdots,0,t)\in R^{q+m}$, t is parametric vector, then $V^+(t_0;t)\leqslant S(T,\overline{x})<+\infty$.

(2) If $\wedge_1(\overline{x})\neq\phi$ or $\lambda T\leqslant 0$ forany $\lambda\in\wedge_0(x)$ then:

$$dV(t_0;t)\leqslant S(T,\overline{x}).$$

(3)
$$\partial V(t_0)\subseteq\wedge_1(\overline{x}).$$

Proof (1) $S(T,x)<+\infty$ is obtained by lemma 4.1 and since $K\cap L(S,T)\neq\phi$ for any $S>S(T,x)$, there is $v=(v_1,v_2)\in T_C(\overline{x})\times T_{R^m}(y)$ such that:

$$\begin{aligned}&f^0(\overline{x};v)<S\\&g_i^0(\overline{x},\overline{y};v_1,v_2)<0, && i\in I(\overline{x},\overline{y})\\&\nabla h_j(\overline{x},\overline{y};v_1,v_2)=0, && p+1\leqslant j\leqslant q\\&-\langle e_l,v\rangle+t_l+0, && 1\leqslant l\leqslant m.\end{aligned}\tag{4}$$

Since $\langle\nabla H(\overline{x},\overline{y}),v\rangle=\langle\nabla_x H(\overline{x},\overline{y}),v_1\rangle+\langle\nabla_y H(\overline{x},\overline{y}),v_2\rangle$ and $\overline{x}\in\mathrm{int}C$, applying lemma 4.2, we get that there is a $\tau_0>0$, such that the equation $H(x,t_0+\tau t)=0$ has a solution on C for $\tau\in(0,\tau_0)$, and the solution is

$$x(\tau)=\overline{x}+\tau\cdot v_1^\tau,\quad\text{where}\quad v_1^\tau\to v\ (\tau\to 0)$$

For
$$g_i^0(\overline{x},\overline{y};v_1,v_2)<0,\qquad i\in I(\overline{x},\overline{y}),$$
$$\limsup_{\tau\downarrow 0}[g_i(x(\tau),t_0+\tau t)-g_i(x,t_0)]/\tau<0,\ i\in I(\overline{x},\overline{y})$$

So while τ is small enough, there is

$$g_i(x(\tau),t_0+\tau t)<0,\qquad i\in I(\overline{x},\overline{y})$$
$$g_i(x(\tau),t_0+\tau t)<0,\qquad i\in\{1,\cdots,p\}-I(\overline{x},\overline{y})$$

and $(x(\tau),t_0+\tau t)$ is a feasible solution of (Pt) for $\tau\in(0,\tau_0)$.

Since $f^0(\overline{x},v_1)<S$,

so $f(x(\tau))-f(\overline{x})<\tau\cdot S$,

$$V(t_0+\tau t)-V(t_0)<\tau\cdot S,\qquad \tau\in(0,\tau_0),$$

therefore, $V^+(t_0;t)\leqslant S$, for any $S>S(T,\overline{x})$,

namely, $V^+(t_0,t)\leqslant S(T,\overline{x})$.

(2) If $\wedge_1(x)\neq\emptyset$ and there is $\lambda\in\wedge_0(\overline{x})$ such that $\lambda\cdot T>0$, because of $\wedge_1(\overline{x})=\wedge_1(\overline{x})+\wedge_0(\overline{x})$, hence $S(T,\overline{x})=+\infty$ and (2) holds.

If $\lambda T\leqslant 0$ for any $\lambda\in\wedge_0(\overline{x})$ and $S(T,\overline{x})<\infty$, for any $S>S(T,\overline{x})$, then $(S,-T)\in\mathrm{cl}(K-L(S,T))$ by lemma 4.1, so for any $\delta>0$, there is $T'=(0,\cdots,0,t')\in R^{q+m}$, $\|T-T'\|<\delta$, $v=(v_1,v_2)\in T_C(\overline{x})\times T_{R^m}(\overline{y})$ such that:

$$\begin{aligned}&f^0(\overline{x};v_1)<S+\delta\\&g_i^0(\overline{x},\overline{y};v_1,v_2)<0, && i\in I(\overline{x},\overline{y})\\&\nabla h_j(\overline{x},\overline{y};v)=0, && p+1\leqslant j\leqslant q\\&-\langle e_l,v\rangle+t_l'=0, && 1\leqslant l\leqslant m\end{aligned}\tag{5}$$

Hence, by the proceeding similar to (1).

$$V^+(t_0;t')\leqslant S(T,\overline{x})+\delta$$

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$$dV(t_0; t') < S(T, \overline{x}) + \delta$$

Let $\delta \to 0$, then $dV(t_0; t) < S(T, x)$.

(3) If $\wedge_1(\overline{x}) = \emptyset$, then the result is obvious, otherwise by (2),

$$dV(t_0; t) < S(T, \overline{x}), \quad \text{for any } t \in R^m.$$

Since $S(T, \overline{x})$ is support function of $\wedge_1(\overline{x})$,

$$\partial V(t_0) \triangleq \{y | \langle y, t \rangle < S(T, \overline{x})\} = \wedge_1(\overline{x}).$$

Directly obtained by theorem 4.1:

Theorem 4.2 (1) Let $M(t_0; t) = \{x \in M(t_0) | \wedge_1(x) \neq \emptyset \text{ or } T \in \wedge_0^0(x)\}$, then:

$$dV(t_0; t) < \inf_{x \in M(t_0; t)} S(T, x).$$

(2) $$\partial V(t_0) \subset \bigcap_{x \in M(t_0)} \wedge_1(x)$$ ■

In order to obtain the bounds of generalized directional derivative $V^{\uparrow}(t_0; t)$ of optimal value function, the programming (Pt) is assumed to satisfy the weak regular condition which is called tameness in [3], [4].

Definition **4.1** If the programming (Pt) satisfies the following property, then (Pt) is called tame at t_0.

there is a compact set A, for every $\varepsilon > 0$, there exist $\delta > 0, a > V(t_0)$, such that the additional constraint distance $(x, A) < \varepsilon$ to (Pt) would not affect the optimal value $V(t)$ in (Pt), when $\|t - t_0\| < \delta$, $V(t) < a$. ■

The tameness is different from the calmness for paramatric perturbation given in [14]. The calmness may essentially be regarded as pointwise lower semi-Lipschitz continuity of the optimal value function of the parametric programming, which the tameness is equivalent to the strict lower semi-continuity of the optimal value function. Some of the properties of thetameness have been described by propposittons 8—10 in [3]. An equivalent form of definition 4.1 is given in [4].

Definition 4.2 If there is $\delta_1 > V(t_0)$, $\delta_2 > 0$, such that (Pt_k) exists the optimal solution x_k, $x_k \to \overline{x} \in M(t_0)$, for any $t_k \to t_0$ satisfies $V(t_k) < \delta_1$, $\|t - t_0\| \delta_2$, then (Pt) is called tameness at t_0.

In the discussion of stability in Section 3, we assume that the set C is a convex compact set. Now it is still assumed. By the definition 4.1:

Property 4.1 Parametric programming (Pt) is tame. ■

Theorem 4.3 Let (Pt) satisfy A-constraint qualification, $x \in M(t_0) \cap \text{int} C$.

(1) If $\lambda T < 0$ for any $\lambda \in \wedge_0(x)$ and $x \in M(t_0)$, then

$$V^{\uparrow}(t_0; t) < \sup_{x \in M(t_0)} S(T, x), \quad \text{for any } t \in R^m.$$

(2) $$\partial V(t_0) \subseteq \text{co}\{\bigcup_{x \in M(t_0)} \wedge_1(x) + \bigcup_{x \in M(t_0)} \wedge_0(x)\}.$$

where co{ } is the convex and closed hull of set {}.

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(3) If $M(t_0)=\{\bar{x}\}$, then $\partial V(t_0)\subseteq\Lambda_1(\bar{x})$.

Proof (1) For any given $\varepsilon>0$, $\eta_k \xrightarrow{\vee} 0$ and $\tau_k\downarrow 0$, by the property 4.1 and def. 4.2, there is an optimal solution x_k belonging to $S(t_0+\eta_k)$ of the programming $P(t_0+\eta_k)$ such that $x_k\to\bar{x}\in M(t_0)$.

For any $t\in R^m$ satifying $\lambda T\leqslant 0$ for any $\lambda\in\Lambda_0(\bar{x})$, if $S(T, x)=+\infty$, then (1) is obviously true, so let $S(T, x)<+\infty$.

For any $S>S(T, \bar{x})$, similar to the proof of theorem 4.1.(2), there exists $t'\in R^m$ satisfying $\|T-T'\|<\varepsilon$, and there is $v=(v_1, v_2)\in T_C(\bar{x})\times T_{R^m}(\bar{y})$ such that:

$$\begin{aligned} &f^0(\bar{x}, v_1)<S+\varepsilon \\ &g_i^0(\bar{x}, \bar{y}; v_1, v_2)<0, && i\in I(\bar{x}, \bar{y}) \\ &\nabla h_j(\bar{x}, \bar{y}; v_1, v_2)=0, && p+1\leqslant j\leqslant q \\ &-\langle e_l, v\rangle+t_i=0, && 1\leqslant l\leqslant m \end{aligned} \tag{6}$$

Similor to the proof of Theorem 4.1. (1), there exists $v_{1k}\to v_1$ $(k\to\infty)$ such that: $H(x_k+\tau_k v_{1k}, t_0+\eta_k+\tau_k t')=0$ and $x_k+\tau_k v_{1k}\in C$ while K is large enough.

By formulas (6): $g_i(x_k+\tau_k v_{1k}, t_0+\eta_k+\tau_k t')<0$, $1\leqslant i\leqslant p$.

so $x_k'+\tau_k\ v_{1k}$ is a feasible solution of $P(t_0+\eta_k+\tau_k t')$ only if K is large enough.

For $f^0(\bar{x}, v_1)<S+\varepsilon$, f^0 is continuous to v_1,

$$f(x_k+\tau_k v_{1k})-V(t_0+\eta_k)<\tau_k(S+\varepsilon).$$

Combining the above and the feasibility of $x_k+\tau_k v_{1k}$,

$$V(t_0+\eta_k+\tau_k t')-V(t_0+\eta_k)\leqslant\tau_k(S+\varepsilon).$$

Therefore, while K is large enough:

$$\inf_{\|t'-t\|<\varepsilon}[V(t_0+\eta_k+\tau_k t')-V(t_0+\eta_k)]/\tau_k\leqslant S+\varepsilon.$$

Considering that $\bar{x}\in M(t_0)$ is not identical with different sequences

$$\limsup_{\tau_k\downarrow 0\ \eta_k\to 0}\ \inf_{\|t'-t\|<\varepsilon}[V(t_0+\eta_k+\tau_k t')-V(t_0+\eta_k)]/\tau_k \leqslant \sup_{x\in M(t_0)} S(T, x)+\varepsilon.$$

By the arbitrariness of ε, we complete the proof of (1).

(2) Let $B=\mathrm{Co}\{\bigcup_{x\in M(t_0)}\Lambda_1(x)+\bigcup_{x\in M(t_0)}\Lambda_0(x)\}$, if B is empty, then $\Lambda_1(x)$ is empty for any $x\in M(t_0)$.

By (1) $V^{\uparrow}(t_0, t)=-\infty$, and $\partial V(t_0)=\varnothing$.

If B is nonempty, by (1):

$$V^{\uparrow}(t_0, t)\leqslant\sup\{\lambda T|\lambda\in B\}\quad \text{for any } t\in R^m.$$

Therefore $\partial V(t_0)\subseteq B$.

(3) By (2) and $\Lambda_1(\bar{x})=\Lambda_1(\bar{x})+\Lambda_0(\bar{x})$ $(\Lambda_1(\bar{x})\neq\varnothing)$. ■

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Reference

[1] A. Auslender. Differentiable stability in non-convex and nondifferentiable programming. Math. Prog. Study 10. 29—41, 1979.

[2] J. Gauvin and F. Dubeau. Differential properities of the marginal function in mathematical programming. Math. Prog. Study 19, 1982.

[3] R. T. Rockafellar, Lagrange multipliers and subderivatives of optimal value functions in non-linear programming. Math. Prog. Study 17, 1982.

[4] B. Gollan. On the marginal function in nonlinear programming. Math. Prog. res. Vol.9. No. 2. 1984.

[5] W. W. Hogan. Point-to-set maps in mathematical programming SIAM. Review Vol. 15, No.1, 1973.

[6] D. Klatte. A. Sufficient condition for lower semicontinuty of solution sets of systems of convex inequalities. Math. Prog. Study 21. 1984.

[7] J.P. Evans. F.J. Gould. Stability in nonlinear programming Oper. res. 18, 1970.

[8] J.Gauvin and J.W. Telle. Differential stability in nonlinear programming. SIAM. J. control and Optim. Vol. 15. No.2, 1977.

[9] F. Lempio and H. Maurer. Differential stability in infinite dimensional nonlinear programming. Appl.Math. Optim. 6, 1980.

[10] A. shapiro. Second-order derivatives of extremal-value functions and optimality conditions for semi-infinite programmes. Math. Oper. res. Vol. 10, No.2, 1985.

[11] R. T. Rockafellar. Marginal values and second-order necessary conditions for optimality. Math. Prog. 26, 1983.

[12] P. Zencke. R. Hettice. Directional derivatives for the value-function in semi-infinite programme. Math. Prog. Vol. 38, 1987.

[13] R. T. Rockafellar. Convex analysis 1970.

[14] F. H. Clarke. Optimization and nonsmooth analysis. 1983.

[15] F. H. Clarke. A new approach to Lagrange multipliers. Math. Oper. res. Vol.1. 1976.

[16] Hiriart-Vrruty. T. B. Refinement of necessary optimality conditions in nondifferentiable programming. Math. Prog. Study. 19, 1982.

参数非光滑优化的微分稳定性

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本文研究了含有向量参数的非光滑优化问题的极值函数或叫做边缘函数的连续性及某种意义下的微分性质．给出了目标函数及不等式约束为李普希兹函数，等式约束为连续可微函数，并且带有闭凸约束集C的非凸非光滑问题的最优值函数的几种方向导数的界，把［4］，［1］中关于一个参数的单边扰动推广到向量参数的扰动，亦可认为是把［2］由光滑函数类推广到李普希兹函数类．

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