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A Note on "Oscillation of Solutions of First Order Delay Differential Equations" *

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Abstract

The aim in this paper is to cite instances in illustration of the incorrectness of certain oscillation theorems in Yan [1]. We also point out that the proof of the main theorems in [1] is not true, and express our correct opinion upon these mistakes.

Ⅰ. Introduction.

In recent years the oscillatory behavior of the first order delay differential equations has been the subject of many investigations. Professor Yan [1] studied the oscillatory behavior of the following nonlinear delay differential equation

$$y'(t)+a(t)y(t)+\sum_{i=1}^{n}p_i(t)\prod_{j=1}^{m_i}|y(t-\tau_{ij}(t))|^{a_{ij}}\operatorname{sgn}y(t-\tau_{ij}(t))=0, \qquad (1.1)$$

where $p_i(t)\geqslant 0, \tau_{ij}(t)>0$ are continuous on $R_+=[0,+\infty)$ and $\tau_{ij}(t)<t$, $\lim_{t\to+\infty}(t-\tau_{ij}(t))=+\infty$, $i\in I=\{1,2,\cdots,n\}$, $j\in J=\{1,2,\cdots,m_i\}$, $a(t)$ is continuous on R_+ and is allowed to take on negative values for arbitrarily large t. The nonnegative numbers a_{ij}, $i\in I$, $j\in J$ satisfy condition $\sum_{j=1}^{m_i}a_{ij}=1$, $i\in I$. Some oscillation or nonoscillation theorems were obtained in [1], one of Yan's main results is the following Theorem A(See [1,Th 3.1]).

Theorem A. Suppose that there are constants τ_0 and a_* such that for sufficiently large T.

$$0\leqslant\sum_{j=1}^{m_i}a_{ij}\tau_{ij}(t)\leqslant\tau_0,\quad i\in I \qquad (1.2)$$

and

$$\inf_{t\geq T}a(t)\geqslant a_*>-\infty \qquad (1.3)$$

If

$$\inf_{t>T,\ \lambda>0}\left\{\frac{1}{\lambda}\sum_{i=1}^{n}p_i(t)\exp[(\lambda+a_*)\sum_{j=1}^{m_i}a_{ij}\tau_{ij}(t)]\right\}>1 \qquad (1.4)$$

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then Eq. (1.1) is oscillatory.

Yan also said that Theorem A prove that the Conjecture of Hunt and Yorke in [2] is true under weaker conditions and include many former results.

The main aim in this paper lies in the following two aspects: First, we show that Theorem A is false by a counterexample; Second, we point out some mistakes in the proof part of Theorem A and express our correct opinion upon Theorem A.

As usual, a solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherewise it is called nonoscillatory. Eq. (1.1) is called oscillatory if al l of it's solutions are oscillatory.

2, Some notes.

In this section we will first cite instances in illustration of the incorrectness of Theorem A, and then point out some mistakes in the proof part of Theorem A and express our correct opinion upon those of Theorem A.

Let us see the following example:

Example Consider the first order differential equation

$$x'(t)+p|x(t-1)|^{\frac{1}{2}}\operatorname{sgn}x(t-1)|x(t-2)|^{\frac{1}{2}}\operatorname{sgn}x(t-2)=0, \tag{2.1}$$

where $p=e^{\frac{3}{2}}$. Obviously, conditions (1.2) and (1.3) hold. On the other hand, from

$$\inf_{\lambda>0}\{\frac{1}{\lambda}p\exp(\frac{3}{2}\lambda)\}=\frac{3}{2}pe>1$$

we know that condition (1.3) holds. Thus, al l conditions of Theorem A are satisfied. But Eq. (2.1) has a nonoscillatory solution $x(t)=-e^{t}$.

In general, if we consider the first order delay differential equation

$$y'(t)+ay(t)+\sum_{i=1}^{n}p_i\prod_{j=1}^{m_i}|y(t-\tau_{ij})|^{a_{ij}}\operatorname{sgn}y(t-\tau_{ij})=0, \tag{2.2}$$

where $p_i>0,\tau_{ij}>0, a_{ij}>0, i\in I, j\in J$ are constants with $\sum_{j=1}^{m_i}a_{ij}=1, i\in I$ and a is a constant. We see then that Eq. (2.2) has always a nonoscillatory solution when $m_i, i\in I$ are even. In fact, set

$$F(\lambda)=\lambda-a+\sum_{i=1}^{n}p_i\exp(\lambda\sum_{j=1}^{m_i}a_{ij}\tau_{ij})$$

Since $F(a)>0$ and $F(-\infty)=-\infty$, we know that there exists a $\lambda_0\in(-\infty,a]$ auch that $F(\lambda_0)=0$. Thus, Eq. (2.2) has a nonoscillatory solution $x(t)=-e^{\lambda_0 t}$.

Now we will point out that the proof part of Theorem A is false. To this end we will repeat certain crucial arguments of the proof part of Theorem A in [1]:

Define a sequence of functions as follows:

$$u_0(t)=\begin{cases}a(t), & t>0,\\ u_0(0), & t<0.\end{cases} \tag{2.3}$$

$$u_k(t)=\begin{cases}\sum_{i=1}^{n} p_i(t)\exp\left(\sum_{j=1}^{m_i} a_{ij}\int_{t-\tau_{ij}(t)}^{t} u_{k-1}(s)ds\right)+a(t), & t>0\\ u_k(0), \quad t<0, \quad k=1,2,\cdots\end{cases}$$

Consider the sequence (2.3) for $t\in R_T=[T,+\infty)$,

$$u_0(t)=a(t)$$

$$u_1(t)=\sum_{i=1}^{n} p_i(t)\exp\left(\sum_{j=1}^{m_i} a_{ij}\int_{t-\tau_{ij}(t)}^{t} a(s)ds\right)+a(t)$$

$$>\sum_{i=1}^{n} p_i(t)\exp\left(a_*\sum_{j=1}^{m_i} a_{ij}\tau_{ij}(t)\right)+a(t)>\lambda_1+a(t)$$

where

$$\lambda_1=\inf_{t\in R_T}\left\{\sum_{i=1}^{n} p_i(t)\exp\left(a_*\sum_{j=1}^{m_i} a_{ij}\tau_{ij}(t)\right)\right\} \tag{2.4}$$

$$u_2(t)>\sum_{i=1}^{n} p_i(t)\exp\left(\sum_{j=1}^{m_i} a_{ij}\int_{t-\tau_{ij}(t)}^{t}(\lambda_1+a(t))ds\right)+a(t)$$

$$>\sum_{i=1}^{n} p_i(t)\exp\left((\lambda_1+a_*)\sum_{j=1}^{m_i} a_{ij}\tau_{ij}(t)\right)+a(t)>\lambda_2+a(t)$$

where

$$\lambda_2=\inf_{t\in R_T}\left\{\sum_{i=1}^{n} p_i(t)\exp\left((\lambda_1+a_*)\sum_{j=1}^{m_i} a_{ij}\tau_{ij}(t)\right)\right\}. \tag{2.5}$$

Using (2.3) and an easy induction, we can prove in general that

$$u_k(t)>\lambda_k+a(t), \quad t\in R_T. \tag{2.6}$$

where

$$\lambda_k=\inf_{t\in R_T}\left\{\sum_{i=1}^{n} p_i(t)\exp\left((\lambda_{k-1}+a_*)\sum_{j=1}^{m_i} a_{ij}\tau_{ij}(t)\right)\right\}, \quad k=3,4,\cdots \tag{2.7}$$

We say that (2.4) and (2.5) are false which are due to:

Firstly, $u_1(t)>\lambda_1+a(t)$ holds only for $t>T$, but it may not hold for $0<t<T$; Secondly, it follows that

$$u_2(t)=\sum_{i=1}^{n} p_i(t)\exp\left(\sum_{j=1}^{m_i} a_{ij}\int_{t-\tau_{ij}(t)}^{t} u_1(s)ds\right)+a(t)$$

$$>\sum_{i=1}^{n} p_i(t)\exp\left(\sum_{j=1}^{m_i} a_{ij}\int_{t-\tau_{ij}(t)}^{t}(\lambda_1+a(s))ds\right)+a(t)$$

holds only when $t-\tau_{ij}(t)>T$, $i\in I$, $j\in J$, but it may not hold for all $t>T$, for example, when $t=T$ since $s\in[T-\tau_{ij}(T),T]$ may not imply that $u_1(s)>\lambda_1+a(s)$, we know that (2.5) may not hold at $t=T$. Similarly, we see that (2.6) may not hold for all $t>T$. (2.6) plays an important role in the late proof of Theorem A, that is, (2.6) guarantees that $\{\lambda_k\}$ is bounded. Hence, the proof of Theorem A is not true.

We also point out that the proof of Lemma 2.2 in [1] is not true. Here we omitt it due to limited space.

In the following, according to the above discussion we will give our correct opinion on Theorem A as follows:

1. Eq. (1.1) should be replaced by one of the following forms:

$$y'(t)+a(t)y(t)+\sum_{i=1}^{n}p_i(t)\prod_{j=1}^{m_i}|y(t-\tau_{ij}(t))|^{a_{ij}}\prod_{j=1}^{2v_i+1}\operatorname{sgn}y(t-\tau_{ij}(t))=0,$$

$$y'(t)+a(t)y(t)+\sum_{i=1}^{n}p_i(t)\prod_{j=1}^{m_i}|y(t-\tau_{ij}(t))|^{a_{ij}}\operatorname{sgn}y(t-\tau_{i1}(t))=0,$$

$$y'(t)+a(t)y(t)+\sum_{i=1}^{n}p_i(t)\prod_{j=1}^{m_i}|y(t-\tau_{ij}(t))|^{a_{ij}}\operatorname{sgn}y(t)=0$$

$$y'(t)\operatorname{sgn}y(t)+a(t)y(t)+\prod_{j=1}^{m_i}|y(t-\tau_{ij}(t))|^{a_{ij}}=0,$$

where v_i is a nonnegative integer with $2v_i+1<m_i$, $i\in I$.

2. Under what additional conditions on $a(t)$, $p_i(t)$, $\tau_{ij}(t)$ and a_{ij} for $i\in I$, $j\in J$ is the sequence $\{\lambda_k\}$ bounded? The correct results will be given in the other paper.

References

[1] Jurang Yan, Oscillation of Solutions of First Order Delay Differential Equations, Nonlinear Analysis, 11(1987), 1279—1287.

[2] Hunt, B. R. & Yorke, J. A., When all solutions of $X'=-\Sigma q_i(t)X(t-T_i(t))$ oscillate, J. Diff. Eqns, 53(1984), 139—145.

关于“一阶时滞微分方程解的振动性”的一个注记

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摘 要

文[1]研究了一阶时滞微分方程

$$y'(t)+a(t)y(t)+\sum_{i=1}^{n}p_i(t)\prod_{j=1}^{m_i}|y(t-\tau_{ij}(t))|^{a_{ij}}\operatorname{sgn}y(t-\tau_{ij}(t))=0. \quad (1)$$

的解的振动性．其中 $p_i(t)\geq 0$, $\tau_{ij}(t)>0$ 在 $R_+=[0,\infty)$ 上连续且 $\tau_{ij}(t)<t$, $\lim_{t\to\infty}(t-\tau_{ij}(t))=\infty$, $i\in I=\{1,2,\cdots,n\}$, $j\in J=\{1,2,\cdots,m_i\}$, $a(t)$ 是 R_+ 上的连续函数且允许对任意大的 t 可取负值．非负数 a_{ij} 满足条件 $\sum_{j=1}^{m_i}a_{ij}=1$, $i\in I$．得到了一些振动和非振动定理．