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Some Characterization of Semi-Fredholm Operators*

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Let $B(H)$ be the set of all linear bounded operators on a Hilbert space H. For $A\in B(H)$, let $R(A)$ and $N(A)$ denote the range and null space of A, respectively. A is said to be semi-Fredholm if $R(A)$ is closed and $\dim N(A)<\infty$ or $\dim N(A^*)<\infty$, where $\dim N$ denotes the dimension of subspace N, A^* is the abjoint of A. Define the minimum modulus of A by

$$r(A)=\inf\{\|Ax\|: \text{dist}(x, N(A))=1\}.$$

It is well known that $r(A)>0$ if and only if $R(A)$ is closed.

In the present note we shall study some characterization of semi-Fredholm operators. It is itself interesting and useful to use for perturbations of generalized inverse. We obtain main results as following.

Theorem 1 For $A\in B(H)$, A is semi-Fredholm if and only if one of the following statements holds.

(1) There is a positive number $\delta>0$ such that $R(E)$ is closed for each $E\in B(H)$ with $\|A-E\|<\delta$.

(2) For every compact operator $K\in B(H)$, $R(A+K)$ is closed.

Theorem 2 If $A\in B(H)$, A is semi-Fredholm if and only if there is a ball $B(A,\varepsilon)\subset B(H)$ with center A and radius ε such that $\lambda_0=\{\lambda: \lambda=\min\{\dim N(E), \dim N(E^*)\}, E\in B(A,\varepsilon)\}$ is bounded.

Proof of Theorems

Proof of Theorem 1 (1) Denote the set of all semi-Fredholm operators in $B(H)$ by $\varphi\mathscr{F}$. It is well known that the set $\varphi\mathscr{F}$ is an open set in $B(H)$. Hence there is a positive number $\delta(A)>0$ for each A, such that E with $\|A-E\|<\delta$ is semi-Fredholm too. Of course, $R(E)$ is closed.

Conversely, for an operator $A\in B(H)$ if A is not semi-Fredholm and $R(A)$ is not closed, then we have nothing to prove. If $R(A)$ is closed, we shall point out that for any $\delta>0$ there is an operator B with $\|A-B\|<\delta$ such that $R(B)$ is not closed. In this case, since A is not semi-Fredholm, we have $\dim N(A)=\dim N(A^*)=\infty$. Suppose that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $N(A)$ and $\{f_i\}_{i=1}^{\infty}$

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is an orthonormal basis of $N(A^*)$ define an operator K by

$$\begin{cases} Ke_i=(1/i)f_i, \\ Kx=0, \quad x\in\{V\{e_i\}_{i=1}^{\infty}\}^{\perp}, \end{cases}$$

where $\{V\{e_i\}_{i=1}^{\infty}\}^{\perp}$ is the orthogonal complement of $V\{e_i\}_{i=1}^{\infty}$. We first show that $R(K)$ is not closed. In fact, it is easy to know that $y_n=\sum_{i=0}^{n}(1/i)f_i\in R(K)$, but $\lim_{n\to\infty}y_n=\sum_{i=1}^{\infty}(1/i)f_i\notin R(K)$. Put $A_\delta=A-(1/2)\delta K$. In this case, it is clear that $R(A_\delta)=R(A)\oplus R(K)$, hence $R(A_\delta)$ is not closed. Observe that $\|A-A_\delta\|<\delta$, then A_δ is as desired.

(2) Observe that the set $\varphi\mathcal{F}$ is invariant under compact perturbation, for any compact operator K, $R(A+K)$ is closed.

On the other hand, if $R(A)$ is not closed, we are done. If A is not semi-Fredholm and $R(A)$ is closed. We can use the operator K as defined in the proof of (1). It is easy to see that K is compact, but $R(A+K)$ is not closed. This completes the proof.

Proof of Theorem 2 "⇒" From Proposition Ⅺ 3.20 (a) in [1], it is evident that $\lambda_0=\min\{\dim N(A), \dim N(A^*)\}$.

"⇐" Assume that for an operator A there is a ball $B(A,\varepsilon)\subset B(H)$ and a constant M such that $\min\{\dim N(B), \dim N(B^*)<M$ for each $B\in B(A,\varepsilon)$.

To prove that A is semi-Fredholm it is enough to prove that $R(A)$ is closed. Let $A=UP$ be the standard polar decomposition of A, assume by way of contradiction that $R(A)$ is not closed, then it is easy to know that 0 is an accumulation point of $\sigma(P)$. If $P=\int\lambda dE_\lambda$ is the spectral decomposition of P, then for $\varepsilon>0$, $\dim(E([0,\varepsilon]))=\infty$. Put $P_\varepsilon=\int_\varepsilon^\infty\lambda dE_\lambda$, and denote $A_\varepsilon=UP_\varepsilon$, it is clear that

$$\|A-A_\varepsilon\|=\|U\int_0^\varepsilon\lambda dE\|<\varepsilon,$$

and $\dim N(A_\varepsilon)=\dim(E([0,\varepsilon])H)=\infty$. This contradicts the assumption. The proof is finished.

Here it is valuable to point out that boundeness of dimension of compact perturbation for semi-Fredholm does not hold.

Next we shall give a result about an operator with closed range.

Proposition 3 Let $A\in B(H)$ be an operator with closed range, then for each operator B with $\|A-B\|<r(A)$,

(1) $\dim R(B)\geq\dim R(A)$;

(2) $\dim N(B)\leq\dim N(A)$.

Proof (1) If $\dim R(B)=\infty$, we are done. Assume that $\dim R(B)=n<\infty$. By the way of contradiction, if $\dim R(B)<\dim R(A)$ and note $\dim R(A)=\dim R(A^*)$, we can choose an orthonomal set $\{e_1, e_2, \cdots, e_n, e_{n+1}\}$ of $R(A^*)$. In this case, $\{Be_i\}_{i=1}^{n+1}$ is linearly dependent, therefore there are complex numbers λ_i, $i=1,2,\cdots,n+1$, which are not all equal to zero and satisfy $\sum_{i=1}^{n+1}\lambda_i Be_i = 0$. Of course, we can choose λ_i such that $\|\sum_{i=1}^{n+1}\lambda_i e_i\| = 1$. In this case,

$$\|A-B\| \geq \|(A-B)(\sum_{i=1}^{n+1}\lambda_i e_i)\| = \|A(\sum_{i=1}^{n+1}\lambda_i e_i)\| \geq r(A),$$

it is a contradiction.

(2) If $\dim N(A) = \infty$, we are done. Next we assume that $\dim N(A) = n < \infty$. By way of contradiction, if $\dim N(B) > \dim N(A)$, then there is an orthonormal set $\{f_1, f_2, \cdots, f_n, f_{n+1}\} \subset N(B)$. If $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal basis of $N(A)$, then the following system of equations

$$\sum_{i=1}^{n+1}\lambda_i(f_i, e_j) = 0, \quad j=1,2,\cdots,n,$$

has a nonzero solution, this shows that there is a nonzero vector $\sum_{i=1}^{n+1}\lambda_i f_i$ which is orthognal to $N(A)$. Choosing λ_i such that $\|\sum_{i=1}^{n+1}\lambda_i f_i\| = 1$, we have

$$\|A-B\| \geq \|(A-B)(\sum_{i=1}^{n+1}\lambda_i f_i)\| = \|A(\sum_{i=1}^{n+1}\lambda_i f_i)\| \geq r(A)$$

a contradiction, the proof is finished.

Reference

[1] Conway, J.E., A course in functional analysis, Springer-Verlag, New York Heidelberg Berlin, 1985.

半Fredholm算子的一些特征

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本文给出了希尔伯特空间中的半Fredholm算子的一些特征，证明了三个算子是半Fredholm算子，当且仅当它的值域在紧扰动和小摄动下均为闭的，并且还证明一个算子是半Fredholm的当且仅当在小摄动中零空间的维数是有界的.