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A Direct Approach to Invariant Density on R^p

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1. Introduction

The need to construct invariant density (ID) on R^p often arises from invariant statistical problem. However there has not been a unified explicit formula for all invariant densities on R^p. Berger (1980[1]) obtained the explicit formula for a very special case of ID on R^p based on haar measure theory. But the case he considered is too restrictive and the treatment he used is not elementary enough as he had hoped. In general, Zhang Yaoting (1986) conjectured that there might be a unified treatment for IDs on R^p since he noted that IDs on R^p were treated individually but similarly by doing some routine computations.

By simply using a well-known elementary change of variables theorem for multiple integrals, this paper deals with ID on R^p by calculus for a quite general case. The sufficient and necessary condition for a Lebesgue measurable function to be an ID on R^p is characterized through a unified equality. With this equality we can draw and verify an ID on R^p very easily. Furthermore, By using this equality this paper obtains the unified explicit formula for so called strictly invariant density (SID) on R^p under transitive action of certain group of transformations, then demonstrates the existence and uniqueness of SID on R^p at this case.

2. Preliminaries

Throughout this paper, We assume

Assumption 2.1 1) $\mathscr{X}$ is a measurable subset of p-dimensional Euclidean space R^p with positive Lebesgue measure and contained in an open subset U of R^p;

2) $f(x)$ is nonnegative measurable on $\mathscr{X}$ with $f(x)<\infty$ a.e. on $\mathscr{X}$;

3) G is a group of one-to-one transformations on U and for every $g\in G$, it holds: i) if $x\in\mathscr{X}$, then $g(x)\in\mathscr{X}$ and ii) g has continuous first partial derivatives in U.

where the concepts of measurable and integration are in the sense of Lebesgue.

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Let $J_g(x)$ denote the absolute value of the Jacobian determimant of the transformation $g\in G$ and $u_f(K)=\int_K f(x)\,dx$ for measurable subset K of U.

Since $g^{-1}\in G$ also has first partial derivatives in U, by chain rule of differentiation we can easily show that $J_g(x)$ does not vanish on U from equality $g\circ g^{-1}=1$. Where 1 denotes identity map on U. Hence following change of variable theorem holds immediately: (see [2], p.421)

Lemma 2.1 For every $g\in G$ and measurable subset K of $\mathscr{X}$, it holds:

$$\int_{g(K)} f(x)dx=\int_K f(g(x))J_g(x)dx \tag{2.1}$$

In fact, the open subset U in assumption 2.1 can usually be taken as $\mathscr{X}$ or R^p. If $U=\mathscr{X}$, the assumption 2.1 can be sufficiently satisfied by following simple condition:

Condition 2.1 i) $\mathscr{X}$ is an open subset of R^p and $f(x)$ as stated in assumption 2.1 2);

ii) G is a group of one-to-one transformations on $\mathscr{X}$ and any $g\in G$ has continuous first parst partial derivatives in $\mathscr{X}$.

3. Invariant density on R^p

Definition 3.1 $f(x)$ is said to be an invariant density on $\mathscr{X}$, if u_f is an invariant measure on $\mathscr{X}$, that is, for every $g\in G$ and measurable subset K of $\mathscr{X}$, $u_f(g(K))=u_f$

Theorem 3.1 $f(x)$ is an invariant density on $\mathscr{X}$ if and only if for every $g\in G$,

$$f(g(x))J_g(x)=f(x) \quad \text{a.e on } \mathscr{X}. \tag{3.1}$$

Proof. For every $g\in G$ and measurable subset K of $\mathscr{X}$, by lemma 2.1,

$$u_f(g(K))=\int_{g(K)} f(x)dx=\int_K f(g(x))J_g(x)dx.$$

By **definition 3.1** $f(x)$ is an invariant density on $\mathscr{X}$ if and only if for every $g\in G$ and measurable subset K of $\mathscr{X}$,

$$u_f(g(K))=u_f(K),$$

equivalently for every $g\in G$, $\int_K f(g(x))\,J_g(x)dx=\int_K f(x)dx$ for all measurable subset K of $\mathscr{X}$, that is, for every $g\in G$, (3.1) holds.

Definition 3.2 $f(x)$ is said to be a strictly invariant density, if for every $g\in G$, it holds:

$$f(g(x))J_g(x)=f(x) \quad \text{for all } x\in\mathscr{X}. \tag{3.2}$$

Obviously SID is ID, on the other hand, note the continuity of $J_g(x)$ by assumption 2.1 3)ii) it follows immediately:

Corollary 3.1 If condition 2.1 holds and $f(x)$ is continuous on $\mathscr{X}$, then $f(x)$ is an invariant density if and only if $f(x)$ is strictly invariant density.

In fact, SID is the most common ID that appears in statistical references.

We shall deal with it in later section. Here we give some examples to show the applications of theorem 3.1, first we state some notations:

$M_{n\times k}(R)$ denotes the set of all matrice of order $n\times k$ with real entries. $\mathrm{GL}(n)=\{A\in M_{n\times k}(R):\det(A)\neq 0\}$ and $A>0$ denotes that A is positive definite real matrix.

Example 3.1 If $\mathscr{X}=\mathrm{GL}(n)$ and g_0: $X\longrightarrow X^{-1}$ $\forall X\in\mathscr{X}$ is the transformation on $\mathscr{X}$, 1 denotes identity map on $\mathscr{X}$.

$G=\{g_0,1\}$ is the group of two transformations g_0 and 1.

Then we have $J_{g_0}(X)=|\det(X)|^{-2n}$ so that (3.1) turns to be $f(X^{-1})\times|\det(X)|^{-2n}=f(X)$ a.e.on $\mathscr{X}$, thus we may consider $f(X)$ of the form $|\det(X)|^{a}$, substitute it in above formula with simplification. We obtain an ID $f(X)=|\det(X)|^{-n}$ immediately.

Example 3.2 $\mathscr{X}=\{(X_1,\cdots,X_p):X_i\in\mathrm{GL}(n),\ X_i>0\ \ i=1,2,\cdots p\}$ $G=\{g_A:A\in\mathrm{GL}(n)\}$ where g_A denotes transformation g_A: $(X_1,\cdots,X_p)\longrightarrow(AX_1A',\cdots,AX_pA')$, $\forall(X_1,\cdots,X_p)\in\mathscr{X}$.

Then we have $J_{g_A}((X_1,\cdots,X_p))=|\det(A)|^{p(n+1)}$ so that (3.1) turns to be

$$f((X_1,\cdots,X_p))=f((AX_1A',\cdots,AX_pA'))|\det(A)|^{p(n+1)}\text{ a.e. on }\mathscr{X},$$

thus we may consider $f((X_1,\cdots,X_p))$ of the form $\prod_1^p|\det(X_i)|^{a_i}$, substitute it in above formula with simplification. We obtain an equivalent formula $|\det(A)|^{p(n+1)+2\Sigma_1^n a_i}=1$ a.e.on $\mathscr{X}$. Hence $f((X_1,\cdots,X_p))=\prod_1^p|\det(X_i)|^{a_i}$ is ID for any fixed array $(a_1,\cdots,a_p)\in R^p$ which satisfies $\Sigma_1^p a_i=-p(n+1)/2$.Specially $f((X_1,\cdots,X_p))=\prod_1^p|\det(X_i)|^{-(n+1)/2}$ is one of IDs on $\mathscr{X}$.

Example 3.3 $\mathscr{X}=\{T=(t_{ij})_{n\times n}\in M_{n\times n}(R):\ t_{ii}>0,t_{ij}=0\ i<j,i,j=1,\cdots,n\}\triangleq T^+$.

$G=\{g_A:\ A\in T^+\}$ where g_A denotes transformation g_A: $T\longrightarrow AT$, $\forall T\in T^+$.

Then we have $J_{g_A}(T)=\prod_1^n a_{ii}^i=t(A)$ where $A=(a_{ii})_{n\times n}\in T^+$ so that (3.1) turns to be

$$f(AT)t(A)=f(T)\ \text{ a.e on } T^+.$$

Note $t(T_1T_2)=t(T_1)t(T_2)$ holds for any two matrice $T_1,T_2\in T^+$, thus we obtain an ID $f(T)=t(T)^{-1}=\prod_1^n t_{ii}^{-i}$ immediately.

Example 3.4 If for every $g\in G$, it holds: $J_g(x)=1$ for all $x\in\mathscr{X}$. Then an nonnegative measurable function $f(x)$ that can be expressed as a function of a maximal invariant is an ID.

Obviously (3.1) turns to be $f(g(x))=f(x)$ a.e. on $\mathscr{X}$ at this case, hence the desired result follows immediately.

Example 3.5 $\mathscr{X}=\mathrm{GL}(n)$

$G=\{g_p:p\in\mathrm{GL}(n)\}$ where g_p denotes transformation g_p: $X\longrightarrow PXP^{-1}$ $\forall X\in\mathscr{X}$.

Then we have $J_{gp}(X)=1$,by example 3.4,(3.1) turns to be $f(PXP^{-1})=$

$\neq f(X)$ a.e on GL(n), hence any nonnegative measurable function that is invariant under all similar transformations is an ID. Especially let $f_0(X) = |\mathrm{tr}(X)|$., $f_m(X) = |\det(X)|^m$, $m = 1, 2, \cdots$, thus $f_m(X)$, $m = 0, 1, 2, \cdots$ are IDs and they are different from each other.

Remarks 3.1 i) It can be seen from the proof of Theorem 3.1 that Theorem 3.1 holds for any case in which change of variable theorem holds. Thus we need not dwell on stated conditions in Assumption 2.1 much since there are many extended results for change of variables theorem. (See [4], [5], etc.)

ii) Since GL(n) and T^+ can be regarded as open subsets of R^{n^2} and $\mathrm{R}^{n(n+1)/2}$ respectively. Therefore Assumption 2.1 holds for all above examples.

iii) From Example 3.3 and 3.5 we see that SIDs on $\mathscr{X}$ under the same G can be very different from each other.

4. SID under transitive action of certain group

Throughout this section, we assume further G acts transitively on $\mathscr{X}$. For any fixed $x_0 \in \mathscr{X}$ and any $y \in \mathscr{X}$, let $G_y(x_0) = \{g \in G: g(x_0) = y\}$

Definition 4.1 For any given $x_0 \in \mathscr{X}$, Jacobian is said to be conjugately identical at x_0, if for any $g \in G_{x_0}(x_0)$ it holds: $J_g(x)_{x=x_0} = 1$.

Lemma 4.1 For any given $x_0 \in \mathscr{X}$ then Jacobian is conjugately at x_0 if and only if $p(y)_{x_0} \quad J_g(x)^{-1}_{x=x_0}$, $g \in G_y(x_0)$ is a well-defined function for all $y \in \mathscr{X}$.

Theorem 4.1 For any given $x_0 \in \mathscr{X}$, we have:

i) There exists a nonzero strictly invariant density on if and only if $p(y)_{x_0}$ is a well-defined measurable function on $\mathscr{X}$ and $p(y)_{x_0} < \infty$ a.e. on $\mathscr{X}$.

ii) Any strictly invariant density $f(x)$ on $\mathscr{X}$ can be expressed as $f(x) = f(x_0)\, p(x)_{x_0}$ $x \in \mathscr{X}$. Therefore strictly invariant densites on $\mathscr{X}$ (under the same G) are unique up to a multiplicative constant.

By Theorem 4.1 we can determine SIDs on $\mathscr{X}$ (under transitive action of G) through following two steps:

Given $x_0 \in \mathscr{X}$, examine: i) Is Jacobian conjugately identical at x_0? ii) Is $p(x)_{x_0}$ integrable on $\mathscr{X}$? If we have positive answers for both i) and ii), then $c\, p(x)_{x_0}$ are all SIDs on $\mathscr{X}$. Otherwise zero fuction is the unique SID on $\mathscr{X}$.

We left proofs of Theorem 4.1 and Lemma 4.1 to later section. Here we give some examples to show the applications of Theorem 4.1.

Example 4.1 $\mathscr{X} = \mathrm{GL}(n)$, $G = \{g_A: A \in \mathrm{GL}(n)\}$ where g_A denotes transformation $g_A: X \longrightarrow AX$, $\forall X \in \mathscr{X}$.

Choose $X_0 = I$ (identity matrix), then for any $Y \in \mathscr{X}$ we have $g_Y(I) = YI = Y$, that is $g_Y \in G_Y(I)$. Note $J_{g_Y}(X)_{X=I} = |\det(Y)|^n$, thus $p(Y)_I = |\det(Y)|^{-n}$ is a SID on $\mathscr{X}$.

Example 4.2 $\mathscr{X} = \{X \in \mathrm{GL}(n): X > 0\}$, $G = \{g_A: A \in \mathrm{GL}(n)\}$, where g_A denotes transformation $g_A: X \longrightarrow AXA$, $\forall X \in \mathscr{X}$.

Choose $X_0 = I \in \mathscr{X}$, for any $Y \in \mathscr{X}$, let $Y^{1/2}$ denote the positive definite matrix B which satisfies equation $B^2 = Y$. It is clear that $g_{Y^{1/2}}(I) = Y$ and $J_{g_Y}(X)_{X=I} = |\det(Y)|^{(n+1)/2}$. Hence $p(Y)_I = |\det(Y)|^{-(n+1)/2}$ is a SID on $\mathscr{X}$.

Example 4.3 $\mathscr{X} = T^+$, $G = \{g_A: A \in T^+\}$ where g_A denotes transformation g_A: $T \longrightarrow TA$, $\forall T \in T^+$.

Choose $T_0 = I$, for any $T \in T^+$ since $g_T(I) = T$ and $J_{g_T}(X)_{X=I} = \prod_1^n t_{ii}^{n+1-i}$ where $T = (t_{ij})_{n\times n}$, therefore $p(T)_I = \prod_1^n t_{ii}^{i-n-1}$ is a SID on $\mathscr{X}$.

Example 4.4 $\mathscr{X} = \{S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}_{n\times n}: S_{11} \in GL(p), S_{22} \in GL(q)\}$ where $n = p + q$. $G = \{g_T: T \in \mathscr{X}\}$ where g_T denotes transformation $g_T: S \longrightarrow TS$, $\forall S \in \mathscr{X}$.

Choose $S_0 = I$, since for any $S = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \in \mathscr{X}$, we have $g_S(I) = S$ and $J_{g_S}(T)_{T_0} = I = |\det(S_{11})|^n |\det(S_{22})|^q$, therefore $p(S)_I = |\det(S_{11})|^{-n} |\det(S_{22})|^{-q}$ is a SID on $\mathscr{X}$.

Example 4.5 $\mathscr{X} = GL(n)$, $G = \{g_{(A,B)}: A, B \in GL(n)\}$ where $g_{(A,B)}$ denotes transformation $g_{(A,B)}: X \longrightarrow AXB$, $\forall X \in \mathscr{X}$

Choose $X_0 = I$, since for any $Y \in \mathscr{X}$ we have $g_{(Y,I)}(I) = Y$ and $J_{g_{(Y,I)}}(X)$ $(X)_{X_0 = I} = |\det(Y)|^n$, therefore $p(Y)_I = |\det(Y)|^{-n}$ is a SID on $\mathscr{X}$.

In each of above examples we may regard $\mathscr{X}$ as an open subset of appropriate dimensional Euclidean space and easily verify that G acts transitively on $\mathscr{X}$ and Jacobian is conjugately identical at I. Thus we may derive the SID directly. However we have following example:

Example 4.6 $\mathscr{X} = T^+$, $G = \{g_{(A,B)}: A, B \in T^+\}$ where $g_{(A,B)}$ denotes $g_{(A,B)}: T \longrightarrow ATB$ $\forall T = (t_{ij})_{n\times n} \in \mathscr{X}$ $(n > 1)$.

Choose $T_0 = I$, we may easily verify that Jacobian is not conjugately identical at I. Therefore there aren't any nonzero SIDs on $\mathscr{X}$.

Remark 4.1 i) In each of Example 4.1 – Example 4.5, $p(Y)_I$ is obviously measurable since $p(Y)_I$ is continuous. Although similar situation holds for most common case, we may not assert the continuity of $p(y)_{x_0}$ generally from the continuity of all first partial derivatives for g.

ii) Above examples show that Lemma 4.1 and Theorem 4.1 are very useful results to determine all SIDs on $\mathscr{X}$ (under transitive action of G) directly. Furthermore, with the help of Lemma 4.1 and Theorem 4.1 we may determine SIDs on $\mathscr{X}$ under intransitive action of G by considering SIDs on each orbit. Especially it is very easy to give the general formula for the case in which $\mathscr{X}$ is the union of countable measurable orbits. However it is usually more convenient to derive SID from (3.1).

iii) Note that SID in Example 3.3 and that in Example 4.3 are different from each other. Usually they are called left SID and right SID respectively. The results about them are also stated respectively and simi larly (see [1], [3], etc.). However this paper gives the unified results in Theorem 4.1.

Proposition 4.1 Suppose G_1, G_2 are two subgroups of G and both of them act transitively on $\mathscr{X}$. $f_1(x), f_2(x)$ are nonzero strictly invariant densities under er G_1, G_2 respectively. Then there exists a nonzero strictly invariant density on $\mathscr{X}$ under G such that $f_1(x) = cf_2(x)$ on $\mathscr{X}$ for some constant c.

Proof Let $f(x)$ denote a nonzero SID on $\mathscr{X}$ under G. Obviously $f(x)$ is a SID on $\mathscr{X}$ under $G_i, i=1,2$. It follows from Theorem 4.1 ii) that $f(x) = c_i f(x)$ $i=1,2$, for some constants c_1, c_2. These equations certainly imply the desired result.

By proposition 4.1 we may assert immediately from Example 3.3 and 4.3 that there are not any nonzero SIDs on $\mathscr{X}$ in Example 4.6 since groups in Example 3.3 and 4.3 can be regarded as subgroups of the group in Example 4.6 and there is more difference than a multiplicative constant between SIDs in Example 3.3 and 4.3.

5. Proofs of Lemma 4.1 and Theorem 4.1

Proof of Lemma 4.1 The "if part" is obvious from the hypothesis since identity map belongs to $G_{x_0}(x_0)$. For the "only if part" we need only to show that for any $y \in \mathscr{X}$, $g_1, g_2 \in G_y(x_0)$ implies $J_{g_1}(x)_{x=x_0} = J_{g_2}(x)_{x=x_0}$. Note that $g_1(x_0) = g_2(x_0)$ hence $x_0 = g_2^{-1} g_1(x_0)$ therefore we have:

$$1 = J_{g_2^{-1}g_1}(x)_{x=x_0} = \{J_{g_2^{-1}}(g_1(x)) J_{g_1}(x)\}_{x=x_0}$$

$$= \{J_{g_2^{-1}}(u)_{u=g_1(x)} J_{g_1}(x)\}_{x=x_0} = J_{g_2^{-1}}(u)_{u=g_1(x_0)=y} J_{g_1}(x)_{x=x_0}.$$

Similarly

$$1 = J_{g_2^{-1}}(u)_{u=y} J_{g_2}(x)_{x=x_0}.$$

Hence

$$J_{g_1}(x)_{x=x_0} = J_{g_2}(x)_{x=x_0}.$$

Proof of Therem 4.1 i) "if part". Suppose $f(x)$ is a nonzero strictly invariant density on $\mathscr{X}$. Then for any $y \in \mathscr{X}$ and $g_y \in G_y(x_0)$, we have

$$f(g_y(x)) J_{g_y}(x) = f(x), \quad \forall x \in \mathscr{X}.$$

Let $x = x_0$

$$f(g_y(x_0)) J_{g_y}(x)_{x=x_0} = f(x_0).$$

Equivalently

$$f(y) = f(x_0) p(y)_{x_0}.$$

Obviously $f(x_0) \neq 0$, then it follows from above equation that $p(y)_{x_0}$ is a well-defined measurable function on $\mathscr{X}$ since $y \in \mathscr{X}$ and $g_y \in G_y(x_0)$ are arbitrary.

"only if part". We need only to verify that $p(y)_{x_0}$ satisfies (3.2). For any $g \in G$ and $y \in \mathscr{X}$, choose $g_{g(y)} \in G_{g(y)}(x_0)$ and $g_y \in G_y(x_0)$ then $g_{g(y)}(x_0) = g(y) = = g(g_y(x_0))$. Thus

$$p(g(y))^{-1} = J_{g_{g(y)}}(x)_{x=x_0} = J_{gg_y}(x)_{x=x_0} \qquad \text{(by Lemma 4.1)}$$
$$= \{J_g(g_y(x))J_{g_y}(x)\}_{x=x_0} = \{J_g(u)_{u=g_y}(x)J_{g_y}(x)\}_{x=x_0}$$
$$= J_g(u)_{u=g_y(x_0)=y} \quad J_{g_y}(x)_{x=x_0}.$$

Hence

$$p(g(y))_{x_0}J_g(y) = \{J_g(u)_{u=y}J_{g_y}(x)_{x=x_0}\}^{-1}J_g(u)_{u=y}$$
$$= J_{g_y}(x)^{-1}_{x=x_0} = p(y)_{x_0}$$

ii) The desired results follow from the "if part" of i) immediately.

Remark 5.1 This paper shows the relations between invariant densities and Jacobians. It gives a unified direct approach to invariant densities and may be of independent interests of statistics. It may also present an elementary background for readers who are involved in haar measure theory and unable to spent much time on it.

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References

[1] Berger.J. (1980), Statistical Decision Theory, Spinger-Verlag, New York Inc.
[2] Apostol, T. (1974), Mathematical Analysis, 2nd Edition, Addison-Wesley, Reading, MA.
[3] Eaton. M.L. (1983). Multivariate Statistics, John Wiley & Sons, Inc.
[4] Rudin, W. (1974), Real and Complex Analysis, McGraw-Hill.
[5] Guzman, M.D. (1980), The American Mathematical Monthly. Vol, 187. No, 9

求R^p上不变密度的一个直接方法

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摘 要

本文利用一个熟知的多元积分变量变换定理对R^p上的不变密度给出了一个不涉及Haar测度理论的纯微积分处理. 对较一般的情形以一个等式的形式给出了非负Lebesgue 可测函数为不变密度的充要条件, 从而提供了求不变密度的一个直接方法. 进一步还对变换群可迁地作用于集的情形导出了所谓严格不变密度的统一的显式表达, 并由此证明了严格不变密度的存在性与唯一性.