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The Inverse Problem for Asgeirsson's Mean Value Equality*

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Abstract

For the linear partial differential operator with constant coefficients $L_x = \sum_{|\alpha| \leq m} a_\alpha D_x^\alpha$, the sufficient and necessary conditions for all C^m solutions $u(x,y)$ of equation (4) satisfying Asgeirsson's mean value equality (2) are that $L_x = c + a\Delta_x$ where $a(\neq 0)$, c are constants, Δ_x is the Laplacian.

It is wellknown that arbitrary homogeneous differential equations of second order with constant coefficients, if not parabolically degenerate, can always be brought into the form

$$u_{x_1} + \cdots + u_{x_n x_n} = u_{y_1 y_1} + \cdots + u_{y_m y_m} - cu$$

by making a suitable linear transformation of the coordinates and, if necessary, by cancelling an exponential factor. We can also eliminate the coefficient c formally (in case it is positive) by introducing an artificial new variable x_{n+1} and setting $u = ve^{cx_{n+1}}$. The differential equation takes on the form

$$u_{x_1 x_1} + \cdots + u_{x_{n+1} x_{n+1}} = u_{y_1 y_1} + \cdots + u_{y_m y_m},$$

where we write again u instead of v. Moreover, by assuming that the u is independent of certain of the variables x and y, we can, without lose of generallty, write the differential equation in the form

$$\Delta_x u = \Delta_y u$$

i.e.

$$\sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} u_{y_i y_i}. \qquad (1)$$

In 1936, L. Asgeirsson proposed the following wellknown mean value theorem[1].

Theorem 1 For every function u which is a twice continuously differentiable solution of equation (1) troughout the region of x,y-space, we have

$$\frac{1}{W_n} \int_{|x-x_0|=r} u(x, y_0)\, ds_x = \frac{1}{W_n} \int_{|y-y_0|=r} u(x_0, y)\, ds_y, \qquad (2)$$

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where ω_n is sphere area of radius r, dS_x, dS_y the surface element. Equality (2) means that the average for fixed $x(=x_0)$ over a sphere of radius r in y-space is the same as the average for fixed $y(=y_0)$ over a sphere of radius r in x-sp space.

A natural problem is that if we set linear partial differential operator with constant coefficients

$$L_x = \sum_{|\alpha| \leq m} a_\alpha D_x^\alpha \tag{3}$$

instead of Δ_x in (1), does the Asgeirsson mean value equality (2) remain true? Liu [2] answered this problem in part. He proved that for all almost-periodic solutions of equation

$$L_x u(x, y) = L_y u(x, y) \tag{4}$$

equality (2) is true. The main result of the present paper is as follows:

Theorem 2 For the linear partial differential operator with constant coefficients (3), the sufficient and necessary conditions for all C^m solutions $u(x, y)$ of the equation (4) satisfying the mean value equality of Asgeirsson (2) are that $L_x = c + a\Delta_x$ where $a(\neq 0)$, c are constants, Δ_x is the Laplacian.

Proof The sufficiency is a direct result of Asgeirsson mean value theorem. We only prove the necessarity.

Write (3) in the form

$$L_x = c + \sum_{i=1}^{n} b_i D_{x_i} + \sum_{i,\ j=1}^{n} a_{ij} D_{x_i} D_{x_j} + \sum_{3 \leq |\alpha| \leq m} a_\alpha D_x^\alpha \tag{5}$$

where $A = (a_{ij})_{n \times n}$ is symmetric matrix. For necessarity, we must prove that if all C^m solutions u of equation (4) satisfying equality (2), then

$$\begin{cases} b_i = 0, & i = 1, 2, \cdots, n. \\ a_{ii} = a \neq 0, & i = 1, 2, \cdots, n. \\ a_{ij} = 0, \quad i \neq j; & i, j = 1, 2, \cdots, n. \\ a_\alpha = 0, & |\alpha| \geq 3. \end{cases} \tag{6}$$

The following two facts are always used to prove that $u(x, y)$ does not satisfy the mean value equality (2):

1) If $f(x)$ is continuous, nonnegative and nonzero function, then

$$\int_{|x|=r} f(x) dS_x > 0.$$

2) If continuous function $f(x)$ is odd for some x_i, then

$$\int_{|x|=r} f(x) dS_x = 0.$$

The proof of (6) is given in six steps.

1° First we show that $b_i = 0$, $i = 1, 2, \cdots, n$. For convenience, we only prove $b_1 = 0$. Otherwise, let

$$f(x) = (b_2 x_1 - b_1 x_2)^2 - 2b_1^{-1}(a_{11}b_2^2 - a_{12}b_1 b_2 + a_{22}b_1^2)x_1$$

then $u(x, y) = f(x) - f(y)$ satisfes (4), but fails in (2), because of

$$\int_{|x|=r} u(x, 0)dS_x > 0 > \int_{|y|=r} u(0, y)dS_y .$$

2° Then we prove that $a_{ii} = a \neq 0$. If for some i, $a_{ii} = 0$, then equation (4) possesses a solution $u(x, y) = x_i^2 - y_i^2$, but the solution fails in (2). So that $a_{ii} \neq 0$. And if for some i, j, $a_{ii} \neq a_{jj}$, then $u(x, y) = a_{jj}x_i^2 - a_{ii}x_j^2$ is a solution of (4), but

$$\int_{|x|=r} u(x, 0)dS_x = (a_{jj} - a_{ii}) \int_{|x|=r} x_i^2 dS_x \neq 0,$$

$$\int_{|y|=r} u(0, y)dS_y = 0.$$

3° Next we have that $a_{ij} = 0$ $(i \neq j)$. If not, for some i, j, $a_{ij} = a_{ji} \neq 0$, then $u(x, y) = x_i^2 - (a_{ii}x_i x_j)/a_{ij}$ satisfies (4), and fails in (2).

4° According to steps 1°-3°, operator (3) can be written in the form

$$L_x = c + a\Delta_x + \sum a_\alpha D_x^\alpha + \sum_{|\alpha| \leq |\beta| \leq m} a_\beta D_x^\beta \qquad (a \neq 0) \qquad (7)$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ are the least order of derivatives of L_x satisfying $|\alpha| \geq 3$, and at least one α_i is odd. Now we prove that $a_\alpha = 0$.

Let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, k is the largest integer which does not exceed $\frac{|\alpha|}{2}$, if $h > k$, we have $\Delta_x^h x^\alpha = 0$. When

$$2!\, b_1 = 4!\, b_2 = \cdots = (2k)!\, b_k = 1,$$

with a suitable selection of λ, by simple computation, we know that

$$u(x, y) = \lambda x^\alpha - \frac{\alpha!}{2na} a_\alpha |x|^2 + b_1(\Delta_x x^\alpha) y_1^2 + \cdots + b_k(\Delta_x^k x^\alpha) y_1^{2k}$$

satisfies equation (4). And in accordance with $u(0, y) = 0$, we get

$$\int_{|x|=r} u(x, 0)dS_x = \int_{|x|=r} [\lambda x - \frac{\alpha!}{2na} a_\alpha |x|^2] dS_x = \int_{|x|=r} [-\frac{\alpha!}{2na} a_\alpha |x|^2] dS_x$$

$$\int_{|y|=r} u(0, x)dS_y = 0.$$

Hence, the above two equalities imply $a_\alpha = 0$.

5° From step 4°, it is known tkat operator (7) can be written in the following form

$$L_x = c + a\Delta_x + \sum_{|\alpha|=k} a_{2\alpha} D_x^{2\alpha} + \sum_{2k < |\beta| \leq m} a_\beta D_x^\beta \qquad (8)$$

here $k \geq 2$. Now, we want to prove that for any α satisfying $|\alpha|=k$, $a_{2\alpha}=b\frac{k!}{\alpha!}$, where b is a constant. Namely, operator (8) can be written in the form

$$L_x = c + a\Delta_x + b\Delta_x^k + \sum_{2k<|\beta|\leq m} a_\beta D_x^\beta . \quad (9)$$

For this purpose, consider the following formulas

$$D_x^{2\alpha} x^{2\alpha} = (2\alpha)!, \quad \Delta_x^k x^{2\alpha} = \frac{k!\,(2\alpha)!}{\alpha!} \quad (|\alpha|=k).$$

It is not difficult to check that for any α with $|\alpha|=k$, as

$$M = \frac{(2\alpha)!}{2n\alpha}(a_{2\alpha} - \frac{k!}{\alpha!} a_{(2k,0,\cdots,0)})$$

the function

$$u(x, y) = x^{2\alpha} + \frac{1}{2!} y_1^2 \Delta_x x^{2\alpha} + \cdots + \frac{1}{(2k)!} y_1^{2k} \Delta_x^k x^{2\alpha} - M|x|^2$$

is a solution of (4). To get (9) from (8), we now prove that $M=0$. It is not difficult to check that $u(x, y)+M|x|^2$ is a solution of (1), then from Asgeirsson theorem, $u(x, y)+M|x|^2$ satisfies the mean value equality (2). And recall thall that we have assumed $u(x, y)$ satisfying mean value equality (2), therefore, we get

$$\int_{|x|=r} M|x|^2 dS_x = 0,$$

hence $M=0$.

6° In this last step, we prove that $b=0$ in (9). For this purpose, we consider the following linear differential operator with constant coefficients

$$L_{x_1} = \sum_{n_1=3}^{m_1} a_{n_1} D_{x_1}^{n_1} + aD_{x_1}^2 \quad (a \neq 0)$$

Obviously, if there exists an n_1, $3 \leq n_1 \leq m_1$, such that $a_{n_1} \neq 0$, then equation

$$L_{x_1} f(x_1) = 0$$

must possess a solution which is not one of the following equation

$$D_{x_1}^2 f(x_1) = 0 .$$

By assumption that all solutions of equation (4) satisfies the mean value equality (2), we get that for operator (9), all solutions $f(x)$ of equation

$$L_x f(x) = L_y f(x)$$

satisfy the following elliptic mean value equality

$$\frac{1}{\omega_n} \int_{|x-x_0|=r} f(x)\,dS_x = f(x_0)$$

for all x_0 in R^n, where ω_n is sphere area of radius r. So that $f(x)$ must satisfy Laplace equation

$$\Delta_x f(x) = 0 . \quad (11)$$

Hence, all solutions of equation (10) are ones of equation (11). Therefore we get $b=0$ from the above explanation.

The above six steps tells us that from steps 1°-3°, operator (3) must take on the form

$$L_x = c + a\Delta_x + \sum_{3\le|\alpha|\le m} a_\alpha D_x^\alpha .$$

And in $\sum_{3\le|\alpha|\le m} a_\alpha D_x^\alpha$, for derivatives of the least order $\alpha=(\alpha_1, \cdots, \alpha_n)$, not only is there no any odd α_i (by step 4°), but also it is impossible for all a_α to be even (by steps 5°-6°). Hence, all a_α must be zero.

Therefore, operator (3) must be in the form

$$L_x = c + a\Delta_x \quad (a \neq 0).$$

This completes the proof of the Theorem.

References

[1] John, F., Plane waves and spherical means applied partial differential equation, New York, 1955.

[2] Liu Baoping, Acta Math. Sinica, 23(1980) 23-36.

关于Asgeirsson均值等式的反问题

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摘 要

对于常系数线性偏微分算子$L_x=\sum_{|\alpha|\le m} a_\alpha D_x^\alpha$，方程$L_x u(x, y)=L_y u(x, y)$的所有$C^m$解满足Asgcirsson均值等式的充分必要条件是$L_x = c + a\Delta_x$，这里$a(\neq 0)$，c为常数，Δ_x为Laplace算子.