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Interpolation by C^1 Quartic Bivariate Splines*

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Abstract

We establish the interpolation schemes by the space of C^1 bivariate piece-wise quartic polynomials defined on a much more general tiangulation of a connected polygonal domain Ω. These schemes demand the location value and partial derivatives of the function f on the grid points and some other points of triangulation Δ of Ω. At last, we describe the reccurent computing method for interpolant splines.

§ 1 Introduction

Let $\Omega\subset R^2$ be a connected polygonal domain, denote by Δ a triangulation of of Ω and suppose that triangles of Δ are labeled $T^{(1)}, \cdots, T^{(N)}$. Given $0\leqslant r\leqslant d$, we consider the linear space of bivariate splines by

$$S_d^r(\Delta)=\{s\in C^r(\Omega):\ s|_{T^{(i)}}\in P_d,\ i=1,\cdots,N\},$$

where P_d is a $(d+1)(d+2)/2$ dimensional linear space of polynomials of total degree d.

Our aim here is to construct an interpolant S_f concerning function $f\in C^1(\Omega)$ from $S_4^1(\Delta)$.

Given a triangulation Δ of a connected polynogal domain Ω. Let

V_I = number of interior vertices,
V_B = number of boundary vertices,
V = total number of vertices,
E_I = number of interior edges,
E_B = number of boundary edges,
E = number of edges,
N = number of tiangles.

It is well known that

$$E_B=V_B,\quad E_I=3V_I+V_B-3,\quad N=2V_I+V_B-2.$$

According to [4], if an interior vertex has only two edges with different slopes attached to it, we call it a singular vertex, then we have

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Theorem Let σ denote the number of singular vertices in Δ, then dim $S_4^1(\Delta) = 3V_I + 4V_B + E + \sigma = 6V_I + 6V_B - 3 + \sigma$.

2 Some Fundamental Concepts

For the need of application, we sketch B-net techniques here. Each spline $s \in S_d^0(\Delta)$ can be written in the form

$$S(x) = S^{(l)}(x) \quad \text{for } x \in T^{(l)}, \quad l = 1, \cdots, N, \tag{2.1}$$

where $S^{(l)}$ is a polynomial of degree d. Throughout this paper, a superscript in brackets refers to a specific triangle $T^{(l)} \in \Delta$. Each the polynomials $S^{(l)}$ can be written as

$$S^{(l)}(x) = \sum_{i+j+k=d} C_{ijk}^{(l)} \frac{d!}{i!j!k!} \beta_1^i \beta_2^j \beta_3^k \tag{2.2}$$

where $\beta_i = \beta_i^{(l)}$, $i = 1, 2, 3$, are the barycentric coordinates (with respect to the triagle $T^{(l)}$) of the point x. These are defined by the equations

$$x = \sum_{i=1}^{3} \beta_i V_i \quad \text{and} \quad \sum_{i=1}^{3} \beta_i = 1, \tag{2.3}$$

where $V_i = V_i^{(l)} = V_i(T^{(l)})$, $i = 1, 2, 3$, denote the vertices of the triangle $T^{(l)}$ (labeled counterclockwise).

With each Bezier-ordinate $C_{ijk}^{(l)}$ we associate a domain point

$$P_{ijk}^{(l)} = \frac{iV_1^{(l)} + jV_2^{(l)} + kV_3^{(l)}}{d} \qquad i + j + k = d. \tag{2.4}$$

The set of all domain points is denoted by $\mathscr{B}_d(\Delta)$. On each triangle T of Δ there are precisely $\binom{d+2}{2}$ points of $\mathscr{B}_d(\Delta)$ spaced uniformly over T. we identify the Bezier-ordinates $C_{ijk}^{(l)}$ on common edges of triangles. This forces the continuity of a piecewise polynomial function. A given B-net uniquely defines a function in $S_d^0(\Delta)$ and vice versa. In dealing with Bezier ordinates, we always assume that the subscripts satisfy $i + j + k = d$.

Given a vertex $v \in \Delta$, we define the p-th ring around v to be the set

$$R_p(v) = \{P_{d-p,j,k} \in T \in \Delta : v = v_1(T)\}. \tag{2.5}$$

A related concept is the disk of order p around v or p-disk $D_p(v)$ defined by

$$D_p(v) = \{P_{ijk} \in T \in \Delta : v = V_1(T), \ i \geqslant d - p\} = \bigcup_{i=0}^{p} R_i(v). \tag{2.6}$$

Consider a triangulation Δ consisting of 2 triangles: triangle $T^{(1)}$ with vertices v_1, v_2, v_3, and triangle $T^{(2)}$ with vertices v_1, v_2, v_4. In this instance, it is convenient to label the vertices of $T^{(2)}$ clockwise. We are concerned with the smoothness of a piecewise polynomial function across the common edge v_1v_2.

It turns out[5] that $S \in S_d^1(\Delta)$ iff

$$C^{(1)}_{i,j,1}=b_1C^{(1)}_{i+1,j,0}+b_2C^{(1)}_{i,j+1,0}+b_3C^{(1)}_{i,j,1},\ i+j=d-1 \tag{2.7}$$

where b_1, b_2, and b_3 are the barycetric coordinates of v_4 with respect to $T^{(1)}$. The equations (2.7) are crucial in what follows. From now on, we will refer to them simply as the smoothness conditions.

Definition 2.1 Let Δ be a triangulation, T be boundary of Ω. Δ is standard if for each triangle T of Δ, $T\cap\Gamma=\phi$ or $\{v\}$ or $\{e\}$. Throughout this paper we suppose that Δ is standard.

3 Interpolation Scheme I

Definition 3.1 Let Ω be a connected polygonal domain, Δ be a triangulation of Ω. If for each interior point of Δ the $\deg(v)$ is odd, i.e., there are odd edges attached to the interior v then we refer Δ to an odd degree triangulation of Ω.

Corollary 3.2 Let Δ be an odd degree triangulation of Ω, then

$$\dim S^1_4(\Delta)=3V_I+4V_B+E=3V+E_B+E. \tag{3.1}$$

Definition 3.3 Let Δ be a triangulation of Ω, denote by Γ the boundary of Ω. Suppose that $T\in\Delta$ is a triangle, T is refered as vertex-boundary triangle if $T\cap\Gamma=v$ belonging to the vertex set of Δ and T is refered as edge-boundary triangle if $T\cap\Gamma=e$ belonging to the edge set of Δ.

Now we suppose that Δ is an odd degree triangulation of Ω.

Interpolation scheme I

Given $f\in C^1(\Omega)$, we require to find a spline function s_f from $S^1_4(\Delta)$ such that

(i) For each vertex v of Δ, $s_f(v)=f(v)$, $\frac{\partial}{\partial x}s_f(v)=\frac{\partial}{\partial x}f(v)$ and $\frac{\partial}{\partial y}s_f(v)=\frac{\partial}{\partial y}f(v)$.

(ii) For each edge $e=v_1v_2$ of Δ, $s_f(\frac{1}{2}e)=f(\frac{1}{2}e)$, where $\frac{1}{2}e=\frac{1}{2}(v_1+v_2)$, i.e., the midqoint of edge e.

(iii) Let $e_1, e_2, \cdots, e_{E_B}$ be all boundary edges of Δ labeled counterclockwise. Picking arbitraryly a boundary edge $e_{i_0}=v^0_1v^0_2$, $\frac{\partial s_f}{\partial n}(\frac{1}{3}v^0_1+\frac{2}{3}v^0_2)=\frac{\partial f}{\partial n}(\frac{1}{3}v^0_1+\frac{2}{3}v^0_2)$ and $\frac{\partial s_f}{\partial n}(\frac{2}{3}v^0_1+\frac{1}{3}v^0_2)=\frac{\partial f}{\partial n}(\frac{2}{3}v^0_1+\frac{1}{3}v^0_2)$; for any $i\neq i_0-1, i_0$, $\frac{\partial s_f}{\partial n_i}(\frac{1}{2}e_i)=\frac{\partial f}{\partial n_i}(\frac{1}{2}e_i)$, where n and n_i are the normal vectors of edges e_{i_0} and e_i, respectively, pointing to inner domain Ω.

It is obvious that the number of interpolatory conditions is equal to dim $S^1_4(\Delta)$ according to the Corollary 3.2.

Theorem 3.4 The spline interpolant s_f of f vith respect to the interpolation scheme I on $S^1_4(\Delta)$ is existent and unique.

In order to prove Theorem 3.4, we need the following two lemmas

Lemma 3.5 Let Δ be an odd degree triangulation consistion of only one interior vertex v, denote by $v_1, v_2, \cdots, v_n$ (n odd integer) the all boundary vertices of Δ (cf. Figure 3.1). then the following spline interpolant s in $S_2^1(\Delta)$ is existent and unique, while giving the values $s(v)$, $\frac{\partial s}{\partial x}(v)$, $\frac{\partial s}{\partial y}(v)$ and $s(v_i)$ ($i = 1, 2, \cdots, n$).

Proof Let Δ be illustrated in Figure 3.1, $s \in S_2^1(\Delta)$ and s's B-nets are also showed in Figure. Suppose that

$$s(v) = \frac{\partial s}{\partial x}(v) = \frac{\partial s}{\partial y}(v) = s(v_i) = 0, \quad (i = 1, 2, \cdots, n). \tag{3.2}$$

Then we can see that $a, b_i, c_i = 0$ ($i = 1, 2, \cdots, n$). Second, from Lemma 3 of [3], we have all of the d_i's are zero.

In figure 3.2, let T be a triangle with vertices $v_i(x_i, y_i)$, $i = 1, 2, 3$, labeled counterclockwise, $A = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$, the algebraic area of T,

$$P_4(x) = \sum_{i+j+k=4} C_{ijk} \frac{4!}{i!\,j!\,k!} \lambda_1^i \lambda_2^j \lambda_3^k$$

be a quartic polynomial defined on T, where λ_i ($i = 1, 2, 3$) are the barycentric coordinates with the vertices v_i ($i = 1, 2, 3$) of T and denote by $n = (\alpha, \beta)$ the unit normal vector of the edge v_1v_2 pointing to the inner domain of T. Denote $|v_1v_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

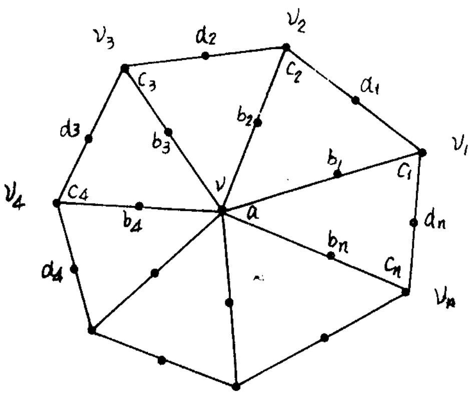

Figure 3.1

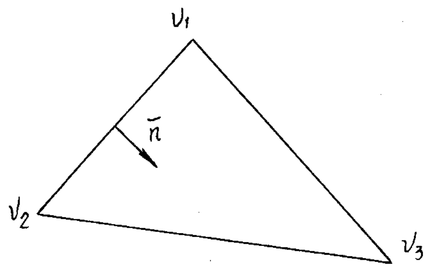

Figure 3.2

Lemma 3.6 (i) $\alpha = (y_1 - y_2)/|v_1v_2|$, $\beta = (x_2 - x_1)/|v_1v_2|$.

(ii)
$$\begin{aligned} \frac{\partial P_4}{\partial n}\Big|_{\lambda_3=0} = \frac{4}{A}\Big[& (\alpha(y_2 - y_3) + \beta(x_3 - x_2)) \sum_{i+j=3} C_{i+1,j,0} \frac{3!}{i!\,j!} \lambda_1^i \lambda_2^j \\ & + (\alpha(y_3 - y_1) + \beta(x_1 - x_3)) \sum_{i+j=3} C_{i,j+1,0} \frac{3!}{i!\,j!} \lambda_1^i \lambda_2^j \\ & + (\alpha(y_1 - y_2) + \beta(x_2 - x_1)) \sum_{i+j=3} C_{ij1} \frac{3!}{i!\,j!} \lambda_1^i \lambda_2^j \Big]. \end{aligned} \tag{3.3}$$

In particular, when $P_4(v_i)=\frac{\partial P_4}{\partial x}(v_i)=\frac{\partial P_4}{\partial y}(v_i)=0 \quad (i=1,2)$, and

$$P_4(\frac{1}{2}v_1v_2)=0, \tag{3.4}$$

then

$$\frac{\partial P_4}{\partial n}\Big|_{\lambda_3=0}=12\frac{|v_1v_2|}{A}(C_{211}\lambda_1^2\lambda_2+C_{121}\lambda_1\lambda_2^2). \tag{3.5}$$

Proof It is obvious.

The proof of Theorem 3.4: The theorem is true iff $s\equiv0$ while the homogeneous interpolation conditions given. First, suppose that the interpolation condittions are homogeneous, then for every point v, we know the all of Bezier ordinates of s_f on $D_1(v)$ are zero for $s_f(v)=\frac{\partial s_f}{\partial x}(v)=\frac{\partial s_f}{\partial y}(v)=0$. For every edge e of Δ, we can specify the Bezier ordinate of s_f on the point $\frac{1}{2}e$ be zero because $s_f(\frac{1}{2}e)=0$. Now we can see that for any triangle T of Δ, all of the Bezier ordinates of s_f on all edges of T are zero.

Second, for an interior vertex v, let us consider $D_2(v)$, denote by Δ the new triangulation of domain cov $[D_2(v)]$, then all of Bezier ordinates of s_f $S_4^1(\Delta)$ on $D_2(v)$ determine a spline $s_{f_v}\in S_2^1(\Delta v)$ such that all of the Bezier odinnates of s_{f_v} same as that of s_f on $D_2(v)$. Thus we have $s_{f_v}(v)=\frac{\partial s_{f_v}}{\partial y}(v)=\frac{\partial s_{f_v}}{\partial y}(v)=0$ and $s_{f_v}(v_B)=0$ for any boundary vertex of Δ_v by first part of proof. According to Lemma 3.5, $s_{f_v}\equiv0$ Hence, if the triangle T of Δ is neither edge-boundary triangle nor vertex-boundary triangle, then $s_f|_T\equiv0$.

Third, from the above two parts of proof, if T is vertex-boundary (an edge-boundary) triangle, then there is only one (are two) Bezier ordinate(s) remaining not to be specified. Let $v_1, v_2, \cdots, v_{E_B}$ be all of the boundary vertices of Δ. No loss generality, Let $i_0=1$ and $T^{(1)}\cap\Gamma=e_{i_0}$, so $T^{(1)}$ is an edge-boundary triangle. Now we label all of edge-boundary and vertex-boundary triangles counterclockwise, denoting by $T^{(1)}, T_1^{(1)}, \cdots, T_1^{(j_1)}, T^{(2)}, \cdots, T^{(i-1)}, T_{i-1}^{(1)}, \cdots, T_{i-1}^{(j_{i-1})}$, $T^{(i)}, \cdots, T^{(E_B)}, T_{E_B}^{(1)}, \cdots, T_{E_B}^{(jE_B)}$ those triangles, where T with lowerscript is the vertex-boundary triangle.

$1'$. It turns out that the two remaining Bezier orcinates of s_f on $T^{(1)}$ are zero according to the interpolation conditions (iii) and Lemma 3.6.

$2'$. If $j_1=0$, then $T^{(2)}$ and $T^{(1)}$ are neighboring triangles. By smoothness condition (2.7) and the interpolation conditions (iii) and Lemma 3.6, we have $S|_{T^{(2)}}\equiv0$. If $j_1\neq0$ the remaining Bezier ordinates of s_f on $T_1^{(m)}$, $C_{211}^{(1),m}$ $(1\leqslant m\leqslant j_1)$ are also zero by the smoothneoss conditions on neighboring eqge, step by step.

$3'$. We have confirmed the all of Bezier ordinates of s_f on all $T_i^{(m)}$ and T $T^{(i)}$ $(i\leqslant E_B-1)$ are zero in $2'$. For $T^{(E_B)}$, if $j_{E_B}=0$, then by the smoothness conditions on the common edge of $T^{(E_s)}$ and $T^{(1)}$, $C_{121}^{(E_s)}=0$, and by the smoothness condition on the common edge of $T_{E_s-1}^{(E_s-1)}$, $C_{211}^{(E_s)}=0$. If $j_{E_s}\neq 0$, similar to 2′, we can see $s_f|_{T_{E_s}^{(E_s)}}\geqslant 0$, hence, the two Bezier ordinates of s_f on $T^{(E_s)}$ are zero.

And now, we can conclude that $s_f\equiv 0$ on Ω.

Denote $s_1: f\mapsto s_f\in S_4^1(\Delta)$ the interpolation operator with respect to the interpolation scheme (I), then

Corollary 3.7 $\forall p\in\Pi_4(\Omega)$, $s_1(p)=p$,

i.e., the interpolation operator s_1 has algebraic preciseness of 4.

4. Interpolation scheme (II)

In the section 3, we discuss an interpolation scheme on an odd degree triangulation Δ of Ω. To deal with the interpolation problems on an arbitrary triangulation Δ of Ω, we want to change Δ to a new odd degree triangulation Δ^* of Ω. Let k be total number of vertices of Δ, the procedure consists of the following.

1*. Choice an edge-boundary triangle T_0 of Δ as the initial triangulation Δ_0. $k=0$.

2*. Arbitrarily choice a vertex from the vertex set of $\Delta\setminus\Delta_k$ such that v neighbors with some vertices $v_1, v_2, \cdots, v_{m_k}$ ($m_k\geqslant 1$) (labeled counterclockwise) (Note: Obviously, the triangle vv_iv_{i-1} and the triangle $vv_{i+1}v_i$ are neighboring triangles of Δ). Calculate the degv_i ($i=1,2,\cdots,m_k$) in Δ_k, when $m_k=2$, goto 5*, $i=2$.

3*. If deg v_i be even, then when $i<m_k$, $i=i+1$ goto 3* and when $i=m_k$ goto 5*;

If deg v_i be odd, then goto 4*.

4*. Choice the barycentric point v_{k_i} of the triangle $vv_{i+1}v_i$ and connect v_{k_i} with v, v_{i+1}, v_i by the line, It turns out three new triangles in $vv_{i+1}v_i$, $i=i+1$, degv_i = degv_i+1.

If $i<m_k$, goto 3*; If $i=m_k$, goto 5*.

5*. $k=k+1$, $\Delta_k=\Delta_{k-1}\cup${all of new points v_{k_i}}$\cup${all of new triangles} If $k<v-3$ goto 2*; If $k=v-3$, define $\Delta^*=\Delta_{v-3}$. end the procedure.

Lemma 4.1 The trianglation Δ^* is an odd degree triangulation of Ω and Δ^* is uniquely determined by the triangulation Δ and the procedure if we choice the most right vertex in $\Delta\setminus\Delta_k$ as v in the procedure 2*.

Now we can discuss the interpolation problem same as that of section 3 on $S_4^1(\Delta^*)$,. But in tha actual applications, Δ is obtained from the information of the location and derivative values of unknown function f, therefore, in general, we don't know the location and derivative values of f at all of new points and the midpoints of new edges on Δ^*. To apply the interpolation scheme (Ⅰ) on Δ^*, we require approximately to calculate all of these information. Here we use the Zienkiewicz's finite element scheme.

Lemma 4.2 Let T be a triangle with vertices $v_i(x_i, y_i)$ $(i=1,2,3)$, showed in Figure 4.1, and v_0 be the barycentric point of T, suppose that $P: f \mapsto \pi_3(T)$ is an interpolation operator which interpolates $f(v_i)$, $\frac{\partial f}{\partial x}(v_i)$ and $\frac{\partial f}{\partial y}(v_i)$ $(i=1,2,3)$ such that for all $q \in \pi_2(T)$, $P(q)=q$, then (all numbers in Figure 4.1 represent relative points).

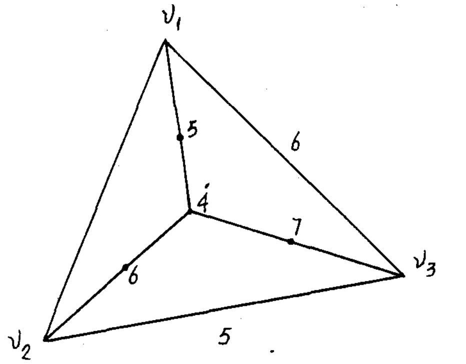

Figure 4.1

$$P(f)(4)=\frac{1}{3}\sum_{i=1}^{3}\Big(f(i)+\frac{1}{3}\frac{\partial f}{\partial x}(i)\big(-x_i+ +\frac{1}{2}x_{i+1}+\frac{1}{2}x_{i+2}\big)+\frac{1}{3}\frac{\partial f}{\partial y}(i)\big(-y_i+\frac{1}{2}y_{i+1}+\frac{1}{2}y_{i+2}\big)\Big] \quad (i \bmod 3)$$

$$\frac{\partial P(f)}{\partial x}(4)=\frac{1}{3}\sum_{i=1}^{3}\frac{\partial f}{\partial x}(i), \qquad \frac{\partial P_{(f)}}{\partial y}(4)=\frac{1}{3}\sum_{i=1}^{3}\frac{\partial f}{\partial y}(i) \tag{4.1}$$

and

$$\begin{aligned}P(f)(i+4)=&\frac{7}{9}f(i)+\frac{1}{9}f(i+1)+\frac{1}{9}f(i+2)\\&+\Big(-\frac{1}{6}x_i+\frac{1}{12}x_{i+1}+\frac{1}{12}x_{i+2}\Big)\frac{\partial f}{\partial x}(i)+\Big(\frac{2}{72}x_i-\frac{3}{72}x_{i+1}+\frac{1}{72}x_{i+2}\Big)\frac{\partial f}{\partial x}(i+1)\\&+\Big(\frac{2}{72}x_i+\frac{1}{72}x_{i+1}-\frac{3}{72}x_{i+2}\Big)\frac{\partial f}{\partial x}(i+2)+\Big(-\frac{1}{6}y_i+\frac{1}{12}y_{i+1}+\frac{1}{12}y_{i+2}\Big)\frac{\partial f}{\partial y}(i)\\&+\Big(\frac{2}{72}y_i-\frac{3}{72}y_{i+1}+\frac{1}{72}y_{i+2}\Big)\frac{\partial f}{\partial x}(i+1)+\Big(\frac{2}{72}y+\frac{1}{72}y_{i+1}-\frac{3}{72}y_{i+2}\Big)\frac{\partial f}{\partial y}(i+2)\\&(i=1,2,3 \text{ and } i, i+1, i+2, \bmod 3).\end{aligned} \tag{4.2}$$

Interpolation Scheme (Ⅱ)

Solute $s_f^* \in S_4^1(\Delta^*)$ such that

(i) For each vertex v and edge e of Δ^*, if v or e is the vertex or edge of Δ respectively, then the interpolation conditions is similar to the (i) and (ii) of scheme (Ⅰ).

(ii) Let v is the new point of Δ^* and v is in the triangle $T=v_1v_2v_3$ of Δ then $s_f^*(v)$, $\frac{\partial s_f^*}{\partial x}(v)$ and $\frac{\partial s_f^*}{\partial y}(v)$ are given by (4.1).

If e_1, e_2, e_3 are the new edges of Δ^* in the triangle $T=v_1v_2v_3$ of Δ, then $s_f^*(\frac{1}{2}e_i)$ $(i=1,2,3)$ are given by (4.2).

(iii) Boundary interpolation conditions are similar to (iii) of (Ⅰ). In particular, such that $e_{l_0}=T_0\cap\Gamma$ where T_0 is the initial triangle in the procedure. Then we have the following results.

Theorem 4.3 The spline interpolant s_f^* of f with respect to the interpolation scheme (Ⅱ) on $S_4^1(\Delta^*)$ is existent and unique, and denote $S_2: f\mapsto s_f^*\in S_4^1(\Delta^*)$, then for any $p\in\pi_2(\Omega)$, $S_2(p)=p$, i.e., the interpolation operator S_2 has only algebraic preciseness of degree 2.

Proof: Theorem 4.3 follows from Theorem 3.4 and Lemma 4.2.

5 Computing Methods of Interpolanes

We only discuss the computing methods for the scheme (Ⅰ)

Let s_f be the spline interpolant of f on $S_4^1(\Delta)$ with regard to the scheme (Ⅰ). We represent s_f as the B-nets on $B_4(\Delta)$. Now we will compute the all of Bezier ordinates of s_f. Let $v(x_0,y_0)$ be a vertex of Δ, a be the Bezier ordinate with respect to v and b_i be the Bezier ordinates with respect to the points $P_i=R_1(v)\cap e_i$ where $e_i=vv_i(x_i,y_i)$, then

Lemma 5.1 Under the above assumptions, we have

$$a=f(v)\ ;\ b_i=f(v)+(x_i-x_0)\frac{\partial f}{\partial x}(v)+(y_i-y_0)\frac{\partial f}{\partial y}(v)\,. \tag{5.1}$$

For each edge $e_{ij}=v_i(x_i,y_i)v_j(x_j,y_j)$, denote by C_{ij} the Bezier ordinate with respect to the point $\frac{1}{2}c_{ij}$. Then we have:

Lemma 5.2

$$C_{ij}=\frac{1}{6}[16f(\tfrac{1}{2}e_{ij})-5f(v_i)-5f(v_j)-(x_i-x_j)\frac{\partial f}{\partial x}(v_i)$$
$$-(y_i-y_j)\frac{\partial f}{\partial y}(v_i)-(x_j-x_i)\frac{\partial f}{\partial x}(v_j)-(y_j-y_i)\frac{\partial f}{\partial y}(v_j)]. \tag{5.2}$$

Let v be an interior point of Δ (cf. Figure 3.1). It leads to the system, according to (2.7),

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -a_1 \\ -a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -a_3 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -a_n & 1 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix}=\begin{bmatrix} \beta_1c_{01}+r_1b_1 \\ \beta_2c_{02}+r_2b_2 \\ \beta_3c_{03}+r_3b_3 \\ \vdots \\ \beta_nc_{0n}+r_nb_n \end{bmatrix}. \tag{5.3}$$

where $v_{i+1}=a_iv_{i-1}+\beta_iv_i+r_iv$.

Define $p_1=\beta_1c_{01}+r_1b_1$, $p_i=\beta_ic_{01}+r_0b_i+a_ip_{i-1}$ $(i=2,\cdots,n)$ (5.4)

then from (5.3), we have

$$d_n=1/(1-a_1a_2\cdots a_n)\cdot p_n \quad\text{and}\quad d_i=p_i+a_1a_2\cdots a_i/(1-a_1a_2\cdots a_n)\cdot p_n$$
$$(i=1,2,\cdots,n).$$

By (5.4)

$$d_i=p_i+a_1a_2\cdots a_i/(1-a_1a_2\cdots a_n)\cdot P_n=\beta_ic_{0i}+r_ib_i+a_ip_{i-1}+a_1a_2\cdots a_i/(1-a_1\cdots a_n)\cdot p_n$$
$$=\beta_ic_{0i}+r_ib_i+a_id_{i-1}\qquad(i=1,2,\cdots,n\ \text{and}\ d_0=d_n) \tag{5.5}$$

Let v's be a boundary vertices of Δ, denote by $v_1,v_2,\cdots,v_{E_B}$ (labeled counetrclockwise). Let $e_{i_0}=v_1v_2$, $T^{(1)}=v_1v_2v_0$, then we specify the two remain ing Bezier ordinates of s_f on $T^{(1)}$.

Now considering $R_2(v_1)$, denote by $v_1'=v_2$, $v_1^2,\cdots$, $v_1^{n_1'}=v_{E_B}$ (labeled counterclo cloch wise) the all of vertices which neighbor with v_1 in Δ (cf. Figure 5.1). Since $s_f|_{T^{(1)}}$ is specified, the $C_{211}^{(1)}$ is known.

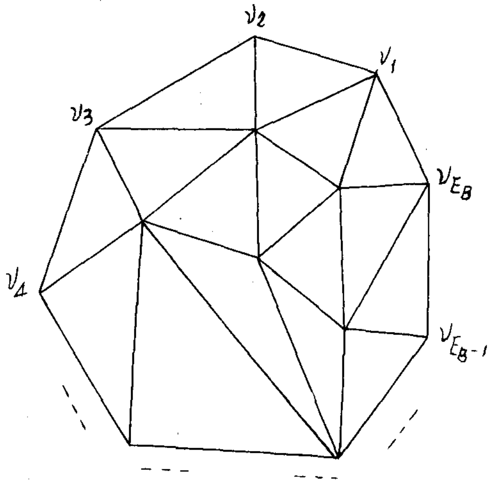

Figure 5.1

If $v_1^{i+2}=a_i^1v_1^i+\beta_i^1v_1^{i+1}+r_i^1v_1$, $(i=1,2,\cdots,n_1-2)$, then

$$d_1^2=a_1^1d_1^1+\beta_1^1c_{11}+r_1^1b_1^1=a_1^1C_{211}^{(1)}+\beta_1^1c_{11}+r_1^1b_1^1$$

$$d_1^i=a_{i-1}^1d_1^{i-1}+\beta_{i-1}^1c_{1i}+r_{i-1}^1b_1^i \quad (i=3,\cdots,n_1-1) \qquad (5.6)$$

On $T^{(E_B)}=v_1v_{E_B}p$, the Bezier ordinate $C_{121}^{(E_B)}=d_1^{n_1-1}$. For $j\,(j=2,\cdots,E_B)$ considering $R_2(v_j)$ we can compute similarly the Bezier ordinates on $R_2(v_j)$ We omit the details.

6. Remarks

(i) For nonstandard triangulation we also may discuss the interpolation problem with respect to the scheme (I) and the scheme (II). Because we can carefully study the Bezier net structure for the other kinds of boundary triangles.

(ii). We can modify the interpolation conditions (iii) of the scheme (I). That is, we give the directional derivative of f on each midpoint of the boundary edge. We can prove that for an odd degree triangulation the spline interpolant s_f is existent and unique if total number of vertex-boundary triangles is odd. And for the nonstandard triangulation it is sufficient that total number of the triangles which have one or three vertices on the boundary Γ of domain Ω. Then we have the following results

Let Δ be an odd triangulation of Ω, Γ be the boundary of Ω, then when total number of the triangles which have one or three vertices on the boundary Γ is odd. the spline interpolant s_f of f with respect to the modified interpolation scheme (I) on $S_4^1(\Delta)$ is existent and unique.

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二元 C^1 四次样条插值

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摘要

本文在较一般的平面三角剖分上构造了一种 C^1 四次样条插值格式．这种格式仅用到被插函数的函数值与一阶导数值信息，并得出插值样条的递推计算格式．