Journal of Mathematical Research and Exposition
Vol.11 No.3 Aug. 1991

On Compositions of Generalized Fractional Integrals and Evaluation of Definite Integrals with Gauss Hypergeometric Functions*

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Abstracts

In this paper, composition formulas for generalized fractional integral operators involving Gauss hypergeometric function are applied to evaluating of definite integrals involving two Gauss hypergeometric functions.

1. Introduction

Our previous paper [1] was devoted to investigating of compositions for general fractional integrals

$$ {}_1I^c_{0+}(a,b)\varphi(x)=\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)}\,{}_2F_1\left(a,b;c;1-\frac{x}{t}\right)\varphi(t)dt, \qquad (1)$$

$$ {}_2I^c_{0+}(a,b)\varphi(x)=\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)}\,{}_2F_1\left(a,b;c;1-\frac{t}{x}\right)\varphi(t)dt \qquad (2)$$

involving the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ and two similar right hand sided operators taken over (x,∞). Integrals of these types are of importance in the theory of fractional calculas and the theory of integral and differential equations, see [1] and [2].

The present paper is devoted to application of the above mentioned results to evaluating definite integrals

$$\int_0^1 s^{a-1}(1-s)^{b-1}(1-zs)^{-c}\,{}_2F_1(d,e;f;zs)\,{}_2F_1\left(g,h;k;\frac{(1-s)z}{1-sz}\right)ds \qquad (3)$$

with two Gauss hypergeometric functions. In section 1, we give two integral representations for ${}_2F_1(a,b;c;z)$. In section 2, we consider some special cases. In particular we obtain the well-known integral representation for ${}_2F_1(a,b;c;z)$, see Remark 1.

2. Integral representations for ${}_2F_1(a,b;c;z)$

Let $a_i\,(i=1,2,\cdots,6)$ be any set of complex numbers and $\beta_i\,(i=1,2,\cdots,6)$ be some of their rearrangement such that $\mathrm{Re}(a_1+a_2-\beta_1-\beta_2)>0$ and $\mathrm{Re}(a_3+a_4-\beta_3-\beta_4)>0$. Let $X_{\nu,\delta}$ be the space from [1] where

*Received Oct.23, 1989.

$$r=\max\{0,\max\sum_{j=1}^{k}(\beta_{i_j}-\alpha_{i_j}),\ (i_1,\cdots,i_k)\subset\{1,\cdots,6\}\},$$
$$\delta>\max_{1<i<6}\mathrm{Re}\,\alpha_i-\frac{1}{2}. \qquad (4)$$

Then according to Theorem 4 from [1] we have equations

$$x_1^{\beta_1}I_{0+}^{\alpha_1+\alpha_2-\beta_1-\beta_2}(\alpha_1-\beta_2,\alpha_2-\beta_2)x^{\beta_2+\beta_3-\alpha_1-\alpha_3}{}_1I_{0+}^{\alpha_3+\alpha_4-\beta_3-\beta_4}(\alpha_3-\beta_4,\alpha_4-\beta_4)\times$$
$$\times x^{\beta_4-\alpha_3-\alpha_4}f(x)=x^{\alpha_5}{}_1I_{0+}^{\beta_5+\beta_6-\alpha_5-\alpha_6}(\beta_5-\alpha_6,\beta_6-\alpha_6)x^{\alpha_6-\beta_5-\beta_6}f(x), \qquad (5)$$
$$x^{\beta_1+\beta_2-\alpha_1}{}_2I_{0+}^{\alpha_1+\alpha_2-\beta_1-\beta_2}(\alpha_1-\beta_2,\alpha_1-\beta_1)x^{\beta_3+\beta_4-\alpha_2-\alpha_3}{}_1I_{0+}^{\alpha_3+\alpha_4-\beta_3-\beta_4}(\alpha_3-\beta_4,\alpha_3-\beta_3)\times$$
$$\times x^{-\alpha_4}f(x)=x^{\alpha_5+\alpha_6-\beta_5}{}_2I_{0+}^{\beta_5+\beta_6-\alpha_5-\alpha_6}(\beta_5-\alpha_6,\beta_5-\alpha_5)x^{-\beta_6}f(x), \qquad (6)$$
$$c_1=\alpha_1+\alpha_2-\beta_1-\beta_2,\ a_1=\alpha_1-\beta_2,\ b_1=\alpha_2-\beta_2,\ p_1=\beta_1,\ q_1=\beta_2-\alpha_1-\alpha_2,$$
$$c_2=\alpha_3+\alpha_4-\beta_3-\beta_4,\ a_2=\alpha_3-\beta_4,\ b_2=\alpha_4-\beta_4,\ p_2=\beta_3,\ q_2=\beta_4-\alpha_3-\alpha_4, \qquad (7)$$
$$c_3=\beta_5+\beta_6-\alpha_5-\alpha_6,\ a_3=\beta_5-\alpha_6,\ b_3=\beta_6-\alpha_6,\ p_3=\alpha_5,\ q_3=\alpha_6-\beta_5-\beta_6.$$

Then (5) can be rewritten in the form

$$x^{p_1}{}_1I_{0+}^{c_1}(a_1,b_1)x^{q_1+p_2}{}_1I_{0+}^{c_2}(a_2,b_2)x^{q_2}f(x)=x^{p_3}{}_1I_{0+}^{c_3}(a_3,b_3)x^{q_3}f(x). \qquad (8)$$

In view of (7), the left side of (8) can be rewritten by

$$I=\frac{x^{p_1}}{\Gamma(c_1)\Gamma(c_2)}\int_0^x(x-\tau)^{c_1-1}\tau^{q_1+p_2}{}_2F_1(a_1,b_1;c_1;1-\frac{x}{\tau})d\tau\int_0^\tau(\tau-t)^{c_2-1}\times$$
$$\times t^{q_2}{}_2F_1(a_2,b_2;c_2;1-\frac{\tau}{t})f(t)dt=\frac{x^{p_1}}{\Gamma(c_1)\Gamma(c_2)}\int_0^x t^{q_2}f(t)dt\times$$
$$\times\int_t^x(x-\tau)^{c_1-1}(\tau-t)^{c_2-1}\tau^{q_1+p_2}{}_2F_1(a_1,b_1;c_1;1-\frac{x}{\tau}){}_2F_1(a_2,b_2;c_2;1-\frac{\tau}{t})d\tau.$$

Making the change of variable $t=\tau+s(x-\tau)$, we obtain

$$I=\frac{x^{p_1}}{\Gamma(c_1)\Gamma(c_2)}\int_0^x(x-t)^{c_1+c_2-1}t^{q_2+p_2+q_1}f(t)dt\times$$
$$\times\int_0^1 s^{c_2-1}(1-s)^{c_1-1}(1-(1-\frac{x}{t})s)^{q_1+p_2}\times$$
$$\times{}_2F_1(a_1,b_1;c_1;\frac{(1-s)(1-\frac{x}{t})}{1-s(1-\frac{x}{t})}){}_2F_1(a_2,b_2;c_2;(1-\frac{x}{t})s)ds. \qquad (9)$$

we write the right side of (8) in the form

$$I=\frac{x^{p_3}}{\Gamma(c_3)}\int_0^x(x-t)^{c_3-1}t^{q_3}{}_2F_1(a_3,b_3;c_3;1-\frac{x}{t})f(t)dt. \qquad (10)$$

Making the change of variable $y=x/t$ and taking into attention (9), (10) and the equalities $c_3=c_1+c_2$ and $p_3+q_3=p_1+p_2+q_1+q_2$, we can rewrite (8) in form

$$x^{p_3+q_3+c_3-1}\int_1^\infty K(y)f(\frac{x}{y})\frac{dy}{y}=0,\quad y>1, \qquad (11)$$

where

$$K(y)=(y-1)^{c_3-1}y^{-c_3}[\frac{y^{-p_2-p_1-q_2}}{\Gamma(c_1)\Gamma(c_2)}\int_0^1 s^{c_2-1}(1-s)^{c_1-1}(1-(1-y)s)^{q_1+p_2}\times$$

$$\times {}_2F_1\left(a_1,b_1;c_1;\frac{(1-y)(1-s)}{1-s(1-y)}\right){}_2F_1(a_2,b_2;c_2;(1-y)s)ds$$

$$-\frac{y^{-a_3}}{\Gamma(c_3)}{}_2F_1(a_3,b_3;c_3;1-y)]. \qquad (12)$$

There exists only trivial soiution of the equation (11) in the space $X_{\gamma,\delta}$, see [1]. Therefore $K(y)=0$ and making the change of variable $z=1-y$, we obtain the formula

$${}_2F_1(\beta_5-\alpha_6,\beta_6-\alpha_6;\beta_5+\beta_6-\alpha_5-\alpha_6;z)=\frac{\Gamma(\beta_5+\beta_6-\alpha_5-\alpha_6)(1-z)^{\beta_1-\alpha_5}}{\Gamma(\alpha_1+\alpha_2-\beta_1-\beta_2)\Gamma(\alpha_3+\alpha_4-\beta_3-\beta_4)}\times$$

$$\times\int_0^1 s^{\alpha_3+\alpha_4-\beta_3-\beta_4-1}(1-s)^{\alpha_1+\alpha_2-\beta_1-\beta_2-1}(1-zs)^{\beta_2+\beta_3-\alpha_1-\alpha_2}\times$$

$$\times{}_2F_1(\alpha_3-\beta_4,\alpha_4-\beta_4;\alpha_3+\alpha_4-\beta_3-\beta_4;zs){}_2F_1\left(\alpha_1-\beta_2,\alpha_2-\beta_2;\alpha_1+\alpha_2-\beta_1-\beta_2;\frac{z(1-s)}{1-sz}\right)ds,$$

$$z<0. \qquad (13)$$

By the same way we can find from (6) the second equality

$${}_2F_1(\beta_5-\alpha_6,\beta_5-\alpha_5;\beta_5+\beta_6-\alpha_5-\alpha_6;z)=\frac{\Gamma(\beta_5+\beta_6-\alpha_5-\alpha_6)(1-z)^{\beta_6-\alpha_4}}{\Gamma(\alpha_1+\alpha_2-\beta_1-\beta_2)\Gamma(\alpha_3+\alpha_4-\beta_3-\beta_4)}\times$$

$$\times\int_0^1 s^{\alpha_1+\alpha_2-\beta_1-\beta_2-1}(1-s)^{\alpha_3+\alpha_4-\beta_3-\beta_4-1}(1-zs)^{\beta_3+\beta_4-\alpha_2-\alpha_3}\times$$

$$\times{}_2F_1(\alpha_1-\beta_2,\alpha_1-\beta_1,\alpha_1+\alpha_2-\beta_1-\beta_2;zs){}_2F_1(\alpha_3-\beta_4,$$

$$\alpha_3-\beta_3;\alpha_3+\alpha_4-\beta_3-\beta_4;\frac{z(1-s)}{1-zs})ds,\ z<0. \qquad (14)$$

Using the method of analytic continuation we obtain the following theorem.

Theorem 1 Let $\alpha_i\,(i=1,2,\cdots,6)$ be any set of complex numbers and $\beta_i\,(i=1,2,\cdots,6)$ be some of their rearrangement such that $\mathrm{Re}(\alpha_1+\alpha_2-\beta_1-\beta_2)>0$ and $\mathrm{Re}(\alpha_3+\alpha_4-\beta_3-\beta_4)>0$. If $z\neq 1$, $|\arg(1-z)|<\pi$, then the formulas (13) and (14) hold.

3. Special cases

The equations (13) and (14) can be used for evaluating definite integrals, rewritten as

$$ {}_2F_1(A,B;C;z)=\frac{\Gamma(c)}{\Gamma(D)\Gamma(C-D)}\int_0^1 s^{D-1}(1-s)^{C-D-1}(1-sz)^{-A'}\times $$

$$ \times\,{}_2F_1(A-A',B;D;sz)\,{}_2F_1\Big(A',B-D;C-D;\frac{(1-s)z}{1-sz}\Big)ds, \qquad (17) $$

$\mathrm{Re}C>\mathrm{Re}D>0, z\neq1, |\arg(1-z)|<\pi.$

This formula coincides with the well-known one, see [3, formula (2.4.3)].

2. $a_1=\beta_4, a_2=\beta_3, a_3=\beta_5, a_4=\beta_6\, a_5=\beta_1, a_6=\beta_2$, (13) reduces to

$$ \int_0^1 s^{c-1}(1-s)^{\bar c-1}(1-sz)^{-a}\,{}_2F_1\Big(a,b;c;\frac{(1-s)z}{1-sz}\Big)\,{}_2F_1(\bar a,\bar c-a-\bar a+b;\bar c;sz)ds= $$

$$ =\frac{\Gamma(c)\Gamma(\bar c)}{\Gamma(c+\bar c)}\,{}_2F_1(a+\bar a,b+\bar c-a;c+\bar c;z), $$

$$ \mathrm{Re}c>0, \mathrm{Re}\bar c>0, z\neq1, |\arg(1-z)|<\pi. \qquad (18) $$

3. $a_1=\beta_6, a_2=\beta_4, a_3=\beta_5, a_4=\beta_1, a_5=\beta_2, a_6=\beta_3$, (13) reduces to

$$ \int_0^1 s^{a_3+a_4-a_6-a_2-1}(1-s)^{a_1+a_2-a_4-a_5-1}(1-zs)^{a_5+a_6-a_1-a_2}\times $$

$$ \times\,{}_2F_1\Big(a_1-a_5,a_2-a_5,a_1+a_2-a_4-a_5;\frac{z(1-s)}{1-zs}\Big)\,{}_2F_1(a_3-a_2,a_4-a_2; $$

$$ a_3+a_4-a_6-a_2;sz)ds $$

$$ =\frac{\Gamma(a_1+a_2-a_4-a_5)\Gamma(a_3+a_4-a_2-a_6)}{\Gamma(a_1+a_3-a_5-a_6)}(1-z)^{a_5-a_4}\times $$

$$ \times\,{}_2F_1(a_3-a_6,a_1-a_6;a_3+a_1-a_5-a_6;z). \qquad (19) $$

For example, if $a_1=\frac{5}{2}, a_2=4, a_3=\frac{3}{2}$, $a_4=5, a_5=1, a_6=2$, then

$$ \int_0^1 s^{-\frac12}(1-s)^{-\frac12}(1-zs)^{-\frac72}\,{}_2F_1\Big(\frac32,2;\frac12;\frac{z(1-s)}{\ldots}\Big)\,{}_2F_1(\ldots 5 \ldots 1 \ldots sz)ds $$