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Complexity of Certain Toeplitz Linear Systems and Polynomial Division*

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Abstract

In this paper, we present a fast algorithm for banded Toeplitz linear systems as as well as banded triangular Toeplitz linear systems. The methods presented in this paper are based on the fast solution of triangular Toeplitz linear systems and the fast solution of Toeplitz linear systems. As an application, we give an algorithm for the polynomial divison with a remainder.

I. Introduction

A Toeplitz matrix is such that its elements t_{ij}'s are function of $(j-i)$, i.e. $t_{ij}=t_{j-i}$. Hence, a Toeplitz matrix is completely defined by its first row and first column. Here we consider banded upper triangular Toeplitz matrices (i.e. $t_k=0$ for $k<0$ and $k>p$) anh banded Toeplitz matrices (i.e. $t_k=0$ for $k>q$ and $k<-p$). Thus, an $n\times n$ banded upper triangular Toeplitz matrix U_b is of the form

$$U_b=\begin{bmatrix} t_0 & t_1 & \cdots & t_p & & \\ & t_0 & \ddots & & \ddots & \\ & & \ddots & \ddots & & t_p \\ & & & & \ddots & \vdots \\ & & & & \ddots & t_1 \\ & & & & & t_0 \end{bmatrix} \tag{1.1}$$

and an $n\times n$ banded Toeplitz matrix T_b is of the form

$$T_b=\begin{bmatrix} t_0 & t_1 & \cdots & t_q & & & \\ t_{-1} & \ddots & \ddots & & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & t_q \\ & \ddots & \ddots & \ddots & \ddots & & \vdots \\ t_{-p} & & \ddots & \ddots & \ddots & \ddots & t_1 \\ & \ddots & t_{-p} & \cdots & \ddots & t_{-1} & t_0 \end{bmatrix} \tag{1.2}$$

Sparse Toeplitz or nearly Toepltz occurs in many mathematical applications Examples are in image processing [9], solution of certain partial differential equation [10], higher order spline approximation [11] and possibly in many

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other applications . In a large number of these situation it is desirable to solve the following linear equation

$$T X = b \tag{1.3}$$

where T is an $n \times n$ Toeplitz matrix. When T is strongly nonsingular, an algor-ithm, presented by Zohar [6], showed that linear equations (1.3) can be solved in $3n^2$ arithmetic operations vs. $O(n^3)$. Further improved results [7] reduced the computational cost from $O(n^2)$ to $O(n \cdot \log^2 n)$. When T is triangular Toeplitz matrixes the number of operations is reduced to $O(n \cdot \log n)$ [5]. Jain [1] presented an algorithm for solving linear systems (1.3), when T is banded His methods are based on certain circular decompositions of the matrix T_b that reduces the solution of $T_b X = b$ to the solution of two sets of simultaneous equations. One set requires solving an $n \times n$ circular linear system. The dom-inant work is in performing the fast Fourier transform. The other set requires solving a Toeplitz equations of order $O(m)$. Therefore, his algorithm requirs $O(n \cdot \log n + m \log^2 m)$ operations by fast solver of Toeplitz linear system, where $m = \max(p, q)$, However, the method is not efficient for banded Toeplitz equations, when $\min(p, q) \ll n$. Another algorithm for banded Toeplitz linear equations was given by Chen [2] when Toeplitz matrices are symmetic. The method is based on the circulant factorization of banded circulant matrices and all rootsfounding of the polynomial whose coefficients are the elements of banded Toeplitz matrices. It is expensive. Recently, Trench [3] presented a new algorithm for the same euqation. The cost of his algorithm is $O(n \max(p, q) + (\min(p, q))^2)$ operations.

Here we show that the computational cost can be reduced to $O(n \cdot \log p)$ and $O(n \cdot \log(q + p) + \min(p, q) \cdot \log^2(\min(p, q)))$ for banded upper triangular Toep-litz linear systems and for banded Toeplitz linear systems respectively. The usefulness of the present methods will clearly be substantial when $\min(p, q)$ is of a lower order than n, e·g. in large scale, sparse, Toeplitz systems. The method presented in this paper is based on the fast solution of triangular Toeplitz linear systems, the fast solution of Toeplitz linear systems and the fast Toeplitz matrix-vector product.

As an application of the method presented in the paper, we considered the polynomial division with a remainder in section 5. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{i=0}^{n} b_i x^i$ be two polynomials of degree m and n respectively ($n \leq m$). We showed that the ceofficients of the quotient $q(x) = \sum_{i=0}^{k} q_i x^i$ and of the remainder $r(x)=\sum_{i=0}^{n-1} r_i x^i$ such that

$$f(x)=g(x)q(x)+r(x)$$

can be found out in $O(m\cdot\log(\min(n,k)))$ arithmetic operations, where $k=m-n$.

2. Toeplitz Matrix-Vector Product and Solution of Triangular Toeplitz Linear Systems

For our purpose, it is necessary to state the following results.

Proposition 2.1 [5] Let T be a square Toeplitz matrix of order n and be a complex n-vector. Then the matrix-vector product Tb can be computed in $O(n\cdot\log n)$ operations.

Let U be an upper triangular Toeplitz matrix which is defined by its first row $(t_0\ t_1\ \cdots\ t_{n-1})$ and L be a lower triangular Toeplitz matrix which is defined by it first column $(b_0\ b_1\ \cdots\ b_{n-1})^T$. The following theorem is efficient to invert U and L if U and L are nonsingular.

Theorem 2.2 [8] Let $Y=(\eta_{n-1}\ \eta_{n-2}\ \cdots\eta_0)^T$ be the solution of the upper triangular Toeplitz linear system

$$UY=e_n \tag{2.1}$$

and $Z=(\beta_0\ \beta_1\ \cdots\ \beta_{n-1})^T$ be the solution of the lower triangular Toeplitz linear system

$$LZ=e_i\ . \tag{2.2}$$

Then the inverse of U and L are given by

$$U^{-1}=\begin{bmatrix}\eta_0 & \eta_1 & \cdots & \eta_{n-1}\\ & \ddots & \ddots & \vdots\\ & \eta_0 & \ddots & \vdots\\ & & \ddots & \eta_1\\ & & & \eta_0\end{bmatrix} \tag{2.3}$$

and

$$L^{-1}=\begin{bmatrix}\beta_0 & & & \\ \beta_1 & \ddots\ \beta_0 & & \\ \vdots & \ddots & \ddots & \\ \beta_{n-1} & \cdots & \beta_1 & \beta_0\end{bmatrix} \tag{2.4}$$

where $e_n=(0\ \ 0\cdots 0\ \ 1)^T$ and $e_1=(1\ \ 0\cdots 0)^T$.

For conveniece, we assume that $n=2^p$ for some positive integer p and denote the 2^k-subvector of y with the last 2^k entries of y as its components by $y^{[k]}$ i.e.

$$y^{(k)}=(\eta_{2^k-1}\cdots\eta_1\quad \eta_0)^T \qquad k=1,2,\cdots,\ P\ , \tag{2.5}$$

which satisfies the linear systems

$$U^{(k)} Y^{(\ )} = (0 \quad 0 \cdots 1)^T \quad , \tag{2.6}$$

where $U^{(k)}$ is submatrix of U of the form

$$U^{(k)} = \begin{bmatrix} t_0 & t_1 & \cdots & t_{2^k-1} \\ & \ddots & & \vdots \\ & t_0 & \ddots & \\ & & \ddots & t_1 \\ & & & t_0 \end{bmatrix} .$$

Clearly

$$U^{(k)} = \begin{bmatrix} U^{(k-1)} & \tilde{U}^{(k-1)} \\ & U^{(k-1)} \end{bmatrix} \qquad Y^{(k)} = \begin{bmatrix} Y_1^{(k-1)} \\ Y^{(k-1)} \end{bmatrix} \tag{2.7}$$

and $U^{(p)} = U$, where

$$\tilde{U}^{(k-1)} = \begin{bmatrix} t_{2^{k-1}} & \cdots & t_{2^{k-1}} \\ \vdots & \ddots \quad \ddots & \vdots \\ t_1 & \cdots \quad \ddots & t_{2^{k-1}} \end{bmatrix}$$

is a $2^{(k-1)} \times 2^{(k-1)}$ Toeplitz matrix and

$$Y_1^{(k-1)} = (\eta_{2^k-1} \cdots \eta_{2^{k-1}})^T .$$

We now consider triangular Toeplitz linear systems

$$U x = d \tag{2.8}$$

We partition the unknown vector x and right hand side d in the way of partitioning Y in (2.5) and (2.7). An efficient algorithm for solving triangular Toeplitz linear systems (2.8) and for inverting U was presented by Chen and Lu [5]

Algorithm I

1. Let $\eta^{(0)} = t_0^{-1}$, $x^{(0)} = t_0^{-1} d_n$

2. For $k = 0, 1, \cdots, p-1$

   i) Compute $Y_1^{(k)} = -(U^{(k)})^{-1} \tilde{U}^{(k)} Y^{(k)}$

   $X_1^{(k)} = (U^{(k)})^{-1} (d_1^{(k)} - \tilde{U}^{(k)} X^{(k)})$

   ii) Assemble the vectors $Y_1^{(k)}$, $Y^{(k)}$ and $X_1^{(k)}$, $X_1^{(k)}$ via (2.7) to generate the result vectors $Y^{(k+1)}$ and $X^{(k+1)}$

   iii) Using $Y^{(k+1)}$ to form $(U^{k+1})^{-1}$

Applying the proposition 2.1 to algorithm I, Chen and Lu [5] showed their results as follows.

Proposition 2.3 Solution of linear systems (2.8) can be computed in $O(n \cdot \log n)$ operations as well as inversion of upper triangular Toeplitz matrices.

Proposition 2.4 Solution of lower triangular Toeplitz linear systems $Lx = d$ can be computed in $O(n \cdot \log n)$ operations as well as inversion of lower triangular Toeplitz matrices.

Corollary 2.5 The matrix-vector product Tb can be computed in $O(\max(n, m)\cdot\log(\min(n,m)))$ operations, where T is an $m\times n$ Toeplitz matrix and b is an n-vector.

Proof Without loss of generality, we assume that $n=km$ for some positive integer k, if $n\geq m$, partitioning T and b as

$$T=(T_1 \quad T_2\cdots T_k)$$

$$b^{*}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix},$$

we obtain that

$$Tb=T_1b_1+T_2b_2+\cdots+T_kb_k,$$

where T_i's are Toeplitz matrices of order m, b_i's are m-vectors, the conclusion is immediate from the proposition 2.1.

The proof is analogous, if $n<m$.

3. Fast Solution of Banded Triangular Toeplitz Linear Systems

In this section, we will present an algorithm for solving the linear system (1.3), when $T=U_b$, i.e.

$$U_bX=b. \tag{3.1}$$

Let $n=pq+p_1$ for some non-negative integers q and $p_1, 0\leq p_1\leq p$. If $p_1\neq 0$, the extension of (3.1) given by

$$U_b^{*}\begin{bmatrix} X \\ Y \end{bmatrix}=\begin{bmatrix} b \\ 0 \end{bmatrix} \tag{3.2}$$

is considered, where U_b^{*} is a banded triangular Toeplitz matrix of order $(q+1)p$ with the first row $(t_0\ t_1\cdots t_p\ 0\ \cdots 0)$, Y is unknown $(p-p_1)$-vector. We partition the matrix U_b^{*} as

$$U_b^{*}=\begin{bmatrix} U_b & U' \\ 0 & U' \end{bmatrix}$$

where T' is $n\times(p-p_1)$ Toeplitz matrix and U' is upper triangular Toeplitz matrix of order p-p_1. Clearly, $Y=0$ and X is the solution of (3.1). Therefore, without loss of generality, we assume that $n=pq$ for some positive integer q in this section.

Partitioning the matrix U_b, unknown vector X and known vector b as

$$U_b=\begin{bmatrix} U_1 & L_1 & & \\ & U_1 & L_1 & \\ & & \ddots & \ddots \; L_1 \\ & & & \ddots \; U_1 \end{bmatrix}, \quad X=\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \end{bmatrix}, \quad b=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_q \end{bmatrix}.$$

We can see the equation (3.1) is equivalant to the following equations

$$U_1 X_q = b_q \ , \tag{3.3-1}$$

$$U_1 X_i + L_1 X_{i+i} = b_i \ , \tag{3.3-2}$$

where U_1 is a $p \times p$ upper triangular Toeplitz matrix of the form

$$U_1 = \begin{bmatrix} t_0 & t_1 & \cdots & t_{p-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t_1 \\ & & & t_0 \end{bmatrix} .$$

L_1 is a $p \times p$ lower triangular Toeplitz matrix of the form

$$L_1 = \begin{bmatrix} t_p & & & \\ t_{p-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ t_1 & \cdots & t_{p-1} & t_p \end{bmatrix} .$$

X_i's are unknown p-subvectors of X and b_j's are p-subvectors of b. Therefore, linear systems (3.1) can be solved in the following stages.

Algorithm Ⅱ (Solver for banded triangular Toeplitz linear systems)

1. For $k=q$, $U \leftarrow U_1$, $d \leftarrow b_q$, perform algorithm Ⅰ
2. For $k=q-1, \cdots, 1$
   i) Compute $L_1 X_{k+1}$
   ii) Compute $\widetilde{b}_k = b_k - L_1 X_{k+1}$
   iii) $U \leftarrow U_1$, $d \leftarrow \widetilde{b}_k$, perform algorithm Ⅰ without computing $y^{(k)}$

It takes q stages to compute the algorithm. For $k=q-1, \cdots, 1$, at each stage the dominant work is in performing triangular Toeplitz matrix-vector multiplication and algorithm Ⅰ without computing $y^{(k)}$, at stage $k=q$, the computation is in performing algorithm Ⅰ. Using proposition 2.1 and 2.3, performing algorithm Ⅱ requires only $O(qp \cdot \log p) + O((q-1)p \cdot \log p) + O((q-1)p) - O((q-1)p) = O(n \cdot \log p)$ operations at most.

Proposition 3.1 Linear system (3.1) can be solved in $O(n \cdot \log p)$ operations as well as lower triangular Toeplitz linear system.

4. Solution of Banded Toeplitz Linear Systems

Using previous results presented in this paper. we consider another case of (1.3) when $T = T_b$, i.e. the banded Toeplitz linear systems

$$T_b X = b \tag{4.1}$$

in this section, where T_b is a banded Toeplitz matrix given by (1.2).

We assume $p \geq q$ without loss of generality. Let $\widetilde{T}_b$ be the $(n+p) \times (n+p)$ upper triangular Toeplitz matrix whose T_b is a submatrix. The following shows the relations between T_b and $\widetilde{T}_b$

$$T_b = \begin{bmatrix} \begin{bmatrix} \widetilde{U} \\ 0 \end{bmatrix} & T & \\ 0 & (0 & U) \end{bmatrix} \qquad (4.2)$$

where $\widetilde{U}$ is a $p \times p$ upper triangular Toeplitz matrix of the form

$$\widetilde{U} = \begin{bmatrix} t_p & \cdots & t_1 \\ & \ddots & \vdots \\ & & t_p \end{bmatrix}$$

Note that X is a solution of (4.1) if, and only if there exists a p-vector Y such that linear equation

$$\widetilde{T}_b \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} b \\ y \end{bmatrix}, \qquad (4.3)$$

has a solution in which $z = 0$, in fact, in this case we have $w = x$, where z and y are p-vectors.

Since $t_p \neq 0$, then $\widetilde{T}_b^{-1}$ is nonsingular. Using Theorem 2.1, the inverse $\widetilde{T}_b^{-1}$ can be computed by solving the following linear equation

$$\widetilde{T}_b u = e_{n+p} . \qquad (4.4)$$

Let $u = (u_{n+p-1}, \cdots, u_0)^T$, then $\widetilde{T}_b^{-1}$ is also an upper triangular Toeplitz matrix of the form

$$\widetilde{T}_b^{-1} = \begin{bmatrix} u_0 & \cdots & u_{n+p-1} \\ & & \vdots \\ & & u_0 \end{bmatrix}.$$

Thus, we can compute $\widetilde{T}_b^{-1}$ by performing algorithm Ⅱ.

Partitioning $\widetilde{T}^{-1}$ as

$$\widetilde{T}_b^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

clearly, we obtain

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} b \\ y \end{bmatrix}, \qquad (4.5)$$

where A is a $p \times n$ Toeplitz matrix B is a $p \times p$ Toeplitz matrix. G is an $n \times n$ upper triangular Toeplitz matrix. D is an $n \times p$ Toeplitz matrix.

It is easy to check that submatrix B is nonsingular. Furthermore, from (1, 5), we can verify the relation between z and y, in fact, we have

$$z = Ab + By. \qquad (4.6)$$

From (4.6), we have that the solution (4.3) fulfills the condition $z = 0$, if and only if

$$By = -Ab. \qquad (4.7)$$

Thus, the linear system (4.1) can be solved in following stages

stages	Operations
1. Compute $\widetilde{T}_b^{-1}$ by performing	$O((n+p) \cdot \log(p+q))$

algorithm I

2. Compute Ab $O(n\cdot\log p)$

3. Solve linear equation (4.7) $O(p\cdot\log^2 p)$

4. Solve linear equation (4.5) $O((n+p)\cdot\log(p+q))$

by performing algorithm I

The evaluation of the arithmetic operations at stages 1 and 4 were attained by proposition 2.4, that at stage 2 was attained by proposition 2.5 and that at stage 3 was attained by the fast solver of Toeplitz linear systems [6]. The overall computational cost is $O(2(n+p)\log(p+q)+n\cdot\log P+P\cdot\log^2 P)=O(n\cdot\log(p+q)+p\cdot\log^2 p)$ opearations.

Proposition 4.1 The banded Toeplitz linear systems (4.1) can be solved in $O(n\cdot\log(p+q)+\min(p,q)\log^2(\min(p,q)))$ arithmetic operations.

5. Polynomial Division

As an application of the method given in section 3. we investigate the classical problem of polynomial division with a remainder in this section.

Let $f(x)=\sum_{i=0}^{m}a_ix^i$, $g(x)=\sum_{i=0}^{n}b_ix^i$ $(a_m\neq 0,\ b_n\neq 0,\ m\geq n)$ be two polynomials over the complex number field, the classical polynomial division is to find out the coefficients of the quotient $q(x)=\sum_{i=0}^{k}q_ix^i$ and of the remainder $r(x)=\sum_{i=0}^{n-1}r_ix^i$ such that

$$f(x)=q(x)g(x)+r(x) \tag{5.1}$$

where $k=m-n$.

It is observed that the problem is equivalent to the following equation

$$\begin{bmatrix} b_n & & & \\ b_{n-1} & b_n & \ddots & \\ \vdots & \ddots & \ddots & b_n \\ b_0 & & \ddots & b_{n-1} \\ & \ddots & & \vdots \\ & & \ddots & \\ & & & b_0 \end{bmatrix} \begin{bmatrix} q_k \\ q_{k-1} \\ \vdots \\ q_0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r_{n-1} \\ \vdots \\ r_0 \end{bmatrix} = \begin{bmatrix} a_m \\ a_m \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}. \tag{5.2}$$

From (5.2), the coefficients of the quotient satisfies the following equation.

$$Q(q_k,\ q_{k-1},\ \cdots,\ q_0)^T=(a_m\ a_{m-1}\cdots a_n)^T \tag{5.3}$$

where Q is a $(k+1)\times(k+1)$ lower triangular Toeplitz matrix of the form.

$$Q = \begin{bmatrix} b_n & & & \\ b_{n-1} & b_n & & \\ \vdots & \ddots & \ddots & \\ b_{n-k} & \cdots & b_{n-1} & b_n \end{bmatrix}$$

where $b_i = 0$ for $i<0$, and the coefficients of the remainder $r(x)$ satisfy that

$$\begin{bmatrix} r_{n-1} \\ \vdots \\ r_0 \end{bmatrix} = \begin{bmatrix} a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} - \begin{bmatrix} b_0 & \cdots & b_{n-1} \\ & & \vdots \\ & & b_0 \end{bmatrix} \begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix}.$$

Therefore, the ploynomial division with a remainder can be done by solving equations (5.3) and (5.1).

It is easy to see that Q is a $(k+1)\times(k+1)$ banded lower triangular Toeplitz matrix with the first column $(b_n\ b_{n-1} \cdots b_0\ \ 0 \cdots 0)^T$, if $k>n$, and

$$\begin{bmatrix} b_0 & \cdots & b_{n-1} \\ & \ddots & \vdots \\ & & b_0 \end{bmatrix} \begin{bmatrix} q_{n-1} \\ \vdots \\ q_0 \end{bmatrix} = \begin{bmatrix} b_{n-k-1} & \cdots & b_{n-1} \\ \vdots & \ddots & \vdots \\ b_0 & \ddots & b_{n-k-1} \\ & \ddots & \vdots \\ & & b_0 \end{bmatrix} \begin{bmatrix} q_k \\ \vdots \\ q_0 \end{bmatrix}$$

if $k<n$. By Proposition 2.4 and 3.1. $(q_k, q_{k-1}, \cdots, q_0)^T$ can be obtained in $O(k\cdot\log(\min(n, k)))$ arithmetic operations via (5.3) and $(r_{n-1}\cdots r_0)^T$ can be obtained in $O(n\cdot\log(\min(n, k))+O(n))$ arithmetic operations via (5.4) by corollary 2.5. The overall computational cost is $O(k\cdot\log(\min(n, k))+n\cdot\log(\min(n, k)))=O(m\cdot\log(n, m-n))$ arithmetic operations.

Proposition 5.1 The coefficients of the quotient $q(x)=\sum_{i=0}^{k} q_i x^i$ and of the remainder $r(x)=\sum_{i=0}^{n-1} r_i x^i$ of the division of polynomials $f(x)=\sum_{i=0}^{m} a_i x^i$ by $g(x)=\sum_{i=0}^{m} b_i x^i$ can be computed in $O(m\cdot\log(\min(n, m-n))$ arithmetic operations. where $a_m, b_n \neq 0$, $m\geq n$, $k=m-n$.

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一类Toeplitz线性方程组与多项式除法的复杂性

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摘　　要

本文给出带状Toeplitz线性方程组，带状三角Toeplitz线性方程组求解的快速方法，其方法基于三角Toeplitz方程与Toeplitz方程的快速求解．并由此给出了一般多次式除法的新算法．