Journal of Mathematical Research and Expostion
Vol. 11. No.4 Nov., 1991

On Amenable Banach Algebras*

Deng Hongjun *Hu Yaohua*

(Hubei University)

Abstract

In this paper we extend the concept of amenable algebra given by Johnson in [3] to n-amenable algebra. We prove several properties which are extension of 1-amenable algebras appeared in [3] and [4], and we give a sufficient condition for a Banach algebra being 2-amenable.

1 Introduction

Johnson ([3]) introduced amenable Banach algebras and proved that the group algebra $L^1(G)$ is amenable if and only if G is amenable, where G is a topological group. J. W. Bunce ([7],[8]) gave the following characterizations of amenable C^*-algebras.

Let A be a C^*-algebra with identity. Then the following three statements are equivalent:

(a) A is amenable

(b) There is a bounded linear map T of $(A\hat{\otimes}A)^*$ into

$$C=\{f\in(A\hat{\otimes}A)^*: af=fa \text{ for all } a\in A\}$$

such that its restriction to C is the identity on C and $T(a\cdot f)=a\cdot T(f)$, $T(f\cdot a)=T(f)\cdot a$, for all $a\in A$, $f\in(A\hat{\otimes}A)^*$.

(c) Let Y be a Banach A-module and X a two-sided A-submodule of Y. Let $f\in X^*$ be such that $f(uxu^*)=f(x)$ for all $x\in X$, $u\in U(A)$. Then there is an $h\in Y^*$ such that h extends f and $h(uyu^*)=h(y)$ for all $y\in Y$, all $u\in U(A)$, where $U(A)$ is the set of units in A.

The involving concepts and signs above will be defined in section 2.

Further, Johnson [4] gave more general equivelent conditions for a Banach algebra to be amenable;

A Banach algebra A is amenable if and only if it has a virtual diagonal.

In this paper we shall introduce n-amenable Banach algebra, left, right n-amenable algebras. We shall give some properties of 2-amenable algebras and

* Received June 4, 1990.

a sufficient condition that a Banach algebra is 2-amenable. And if we take $n=1$, then 1-amenable Banach algebra is Banach amenable algebra introduced by Johnson ([3]). We shall say n-amenable algebra for brevity.

2 Notations

Let A be a Banach space over complex field $\mathbf{C}$. then a Banach space X over $\mathbf{C}$ is a Banach A-module if it is a two-sided A-module and there is a positive real number K such that $\|ax\|\leq K\|a\|\,\|x\|$, and $\|xa\|\leq K\|x\|\,\|a\|$, for all $a\in A$, $x\in X$. Let A be a Banach algebra with unit. Consider the category $\mathscr{A}$ whose objects are A-modules and whose morphisms are continuous homomorphic mappings called below A-module morphisms for A-modules X and Y, the set of morphisms of X into Y is denoted by $\mathrm{Hom}(X,Y)$. while the (generally larger) set of all liner operators from X into Y is denoted by $B(X, Y)$. A morphism $\varphi\in\mathrm{Hom}(X,Y)$ is admissible if its kernel has a Banach complement in X and its image is closed and has a Banach complement in Y. An A-module P is called relatively projective if for any admissible epimorphism $\sigma\in\mathrm{Hom}(X,Y)$, $X,Y\in\mathscr{A}$ and any $\psi\in\mathrm{Hom}(P,Y)$ there is a $\tau\in\mathrm{Hom}(P, X)$ with $\sigma\tau=\psi$. We will say projective rather than relatively projective for brevity. An exact sequence of A-modules

$$\cdots\longrightarrow P_2\xrightarrow{\varphi_1}P_1\xrightarrow{\varphi_0}P_0\xrightarrow{\varepsilon}X\longrightarrow 0$$

is callod a (relatively) projective resolvent of the A-module X if all the P_n, $n=0,1,2,\cdots$ are projective and all the morphisms ε and φ_n, $n=0,1,2,\cdots$ are admissible.

For any $X\in\mathscr{A}$, consider the inductively defined sequence of A-modules $\beta_{n+1}(x)=A\hat{\otimes}\beta_n(X)$, where $\beta_0(X)=A\hat{\otimes}X$. Define ∂_n: $\beta_{n+1}(x)\to\beta_n(x)$, $n=1,2,\cdots$ by the formula

$$\partial_n(a_1\otimes\cdots\otimes a_{n+1}\otimes x)=a_1a_2\otimes a_3\otimes\cdots\otimes a_{n+1}\otimes x-a_1\otimes a_2a_3\otimes\cdots\otimes a_{n+1}\otimes x$$
$$+\cdots+(-1)^{n+1}a_1\otimes\cdots\otimes a_{n-1}\otimes a_na_{n+1}\otimes x+(-1)^{n+2}a_1\otimes\cdots\otimes a_n\otimes a_{n+1}x.$$

Then the sequence

$$\cdots\to\beta_2(X)\to\beta(X)\to\beta_0(X)\to X\to 0$$

where π is the canonical morphism, and ∂_n, $n=0,1,\cdots$ are as defined above, is a projective resolvent of the A-module X. (see [16]).

For any resolvent of X, we will write $\mathscr{P}(X)$. Let $X,Y\in\mathscr{A}$ and $\mathscr{P}(X)$ be a projective resolvent of X. Denote the complex of abelian groups by $\mathrm{Hom}(\mathscr{P}, Y)$

$$0\to\mathrm{Hom}(P_0,Y)\to\mathrm{Hom}(P_1,Y)\to\mathrm{Hom}(P_2,Y)\to\cdots$$

where ψ_n: $\mathrm{Hom}(P_n,Y)\to\mathrm{Hom}(P_{n+1},Y)$ is defined by $[\psi_n(T)](x)=T[\varphi_n(x)]$. The cohomology groups of $\mathrm{Hom}(\mathscr{P}, Y)$ depend only on X, Y but not on the concrete form of the resolvent $\mathscr{P}(X)$ ([16], p401), thus for each pair of A-modules X

and Y it is uniquely defined. A sequence of abelian groups, the cohomology groups of the complex Hom($\mathscr{P}$, Y), denoted henceforth by $Ext^n(X,Y)$, $n=0,1,\cdots$, Clearly $Ext^0(X,Y)=\mathrm{Hom}(X,Y)$. Further, by standard arguments of homological algebra one can proves[16]:

Lemma 2.1 Let $0\to X'\to X\to X''\to 0$ be an exact sequence of A-moduls with all morphism admissible. Then for any A-module Y there is an exact sequence

$$0\to Ext^0(X'',Y)\to\cdots\to Ext^{n-1}(X',Y)\to Ext^n(X'',Y)\to Ext^n(X,Y)$$
$$\to Ext^n(X',Y)\to\cdots$$

Now we can define homological dimension of an A-module X:

Definition 2.1 An A-module X has homological dimension n if $Ext^{n+1}(X,Y)=0$ for any Y but there is a $Z\in\mathscr{A}$ such that $Ext^n(X,Z)\neq 0$, The homological demension of an A-module X will be written as $h.\dim X$.

Definition 2.2 The small global dimension of a Banach algebra A is the number $s\cdot\dim A=\sup\{h\cdot\dim X\}$, where the sup is taken over all finite dimensional A-modules.

Let A be a Banach algebra, and let Y be a normed bimodule over A. An n-cochain ($n=0,1,\cdots$) is a continuous n-linear operator $f: A\times A\times\cdots\times A\to Y$. The n-cochains form an abelian group under addition, denoted by $L^n(A,Y)$. The boundary operator $\delta^n: L^n(A,Y)\to L^{n+1}(A,Y)$ is defined by

$$\delta T(a_1,\cdots,a_{n+1})=a_1T(a_2,\cdots,a_{n+1})-T(a_1a_2,\cdots,a_{n+1})+T(a_1,a_2a_3,\cdots,a_{n+1})$$
$$+\cdots+(-1)^nT(a_1,\cdots,a_{n-1},a_na_{n+1})+(-1)^{n+1}T(a_1,\cdots,a_n)a_{n+1}$$

The equality $\delta^{n+1}\delta^n=0$ is easily verified, hence the cochains form a complex

$$0\to L^0(A,Y)\to L^1(A,Y)\to L^2(A,Y)\to\cdots \tag{2.1}$$

The nth cohomology group $H^n(A,Y)$ of which is called the nth cohomology group of the Banach algebra A with coefficients in the (normed) bimodule Y. We will denote Ker δ^{n+1} by $Z^n(A,Y)$, and Imδ^n by $B^n(A,Y)$. By ([16] lemma 6.1) We have:

Lemma 2.2 For any left normed A-module X and Y we have, up to an isomorphism,

$$Ext^n(X,Y)=H^n(A,B(X,Y))$$

Let X be a Banach A-module, X^* be the dual space of X if we define,

$$(ya,x)=(y,ax),\quad (ay,x)=(y,xa)\quad a\in A,\ x\in X,\ y\in X^*$$

then the dual X^* of X becomes a Banach A-module. The tensor product $A\hat{\otimes}A$ is a Banach A-module if we define

$$a(b\otimes c)=ab\otimes c,\quad (b\otimes c)a=b\otimes ca,\quad a,b,c\in A$$

π will denote the continuous linear map $A\hat{\otimes}A\to A$ defined by $\pi(a\otimes b)=ab$ $(a,$

$b\in A$).

The complex 2.1 can be difined for any Banach A-module Y. In this paper we will restrict us to the case in which the Banach A-moeule Y is the dual of an A-module X.

Definition 2.3 Let A be a Banach algebra. If $H^n(A, X^*)=0$, for any A-module X such that $ax=0$, $a\in A$, $x\in X$, then we will say A is left n-amenable. Similarly, we can define right n-amenable Banach algebra.

Definition 2.4 Let A be a Banach algebra, we say A is n-amenable if $H^n(A, X^*)=0$, for any A-module X.

Standard algebraic formula show ([3],1, a) how, given A, X, n, p, we can find a Banach A-module X_1, with $H^{n+1}(A, X^*)\cong H^n(A, X_1^*)$, thus we have the following.

Corollary 2.3 If a Banach algebra A is n-amenable then A is $n+p$-amenable for all $n\geq 1$, $p\geq 0$.

Definition 2.5 Let A be a Bananch algebra. If $H^n(A, X^*)=0$ for any Banach A-module X and there is a A-module Y such that $H^{n-1}(A, Y^*)\neq 0$, then we will say n is the amenable degree of A.

In lemma 1.2 If we take $Y=\mathbf{C}$, then $B(X, Y)=X^*$, so $Ext^n(X, \mathbf{C})=H^n(A, X^*)$ hence we have

Corollary 2.4 If n is the amenable degree of a Banach algebra A, then $n\leq s.\dim A+1$, i.e., if $k=s.\dim A$, then $H^{n+p}(A, X^*)=0$ for any A-module X, and $p>0$.

3. Fundamental properties of n-amenable algebras

Let A be a Banach algebra, X be a Banach A-modulc and $\widetilde{A}$ is the algebra obtained by adjoining an identity e to A, then X becomes a unital $\widetilde{A}$-module by defining $ex=x=xe$, for all $x\in X$ we denote the $\widetilde{A}$-module by $\widetilde{X}$, then we have ([3],1, d)

$$H^1(\widetilde{A}, \widetilde{X})\cong H^1(A, X)$$

More generally we can easily, show that the same ismorphism between the higher cohomology groups ([11], Theorem 2) is

(a) $$H^n(\widetilde{A}, \widetilde{X})\cong H^n(A, X) \text{ for all } n.$$

Next we have ([3],1, c).

(b) Let A be a Banach algebra with identity element e, if X is a Banach A-module then

$$H^n(\widetilde{A}, \widetilde{X})\cong H^n(A, exe)$$

Lemma 3.1 Let A be any Banach algebra, B be a Banach algebra with an identity element e, hence $A\oplus B$ is a Banach algebra. Let X be an $A\oplus B$-

module. If $T\in Z^m(A\oplus B, X)$, then there exits an $S\in L^{m-1}(A\oplus B, X)$ such that $(T-\delta S)(c_1,\cdots,c_m)=0$ whenever one of the $c_i(\in A\oplus B)$ is equal to e.

Proof we prove this by induction on m. If $m=1$, set $S=eT(e)-T(e)e$

then $$(T-\delta S)(c)=T(c)-cS+Sc=T(c)-ceT(e)+cT(e)e+eT(e)c-T(e)ec$$

since $$T(e)=eT(e)+T(e)e \quad (\text{by } \delta T(e\cdot e)=0)$$

we have $eT(e)e=0$ and so $(T-\delta S)(c)=0$ when c is equal to e.

Now suppose that the lemma has been proved for al l $m\leq n-1$ and consider $T\in Z^n(A\oplus B), X)$, if $\widetilde{T}$ is the corrosponding element of $Z^{n-1}(A\oplus B, L_1(A\oplus B, X))$ ([12].p261) there exists, by our hypothesis, an element $\widetilde{S}\in L^{n-2}(A\oplus B, L^1(A\oplus B, X))$ such that $(T-\delta S)(c_1,\cdots,c_{n-1})=0$, whenever one of the c_i is equal to e. Define $S\in L^{n-1}(A\oplus B, X)$ by setting

$$S(c_1,\cdots,c_{n-1})=(\widetilde{S}(c_1,\cdots,c_{n-2}))(c_{n-1})$$

Then $(T-\delta S)(c_1,\cdots,c_n)=0$ whenever one element of the c_i's $(i=1,\cdots,n-1)$ is equal to e. Now write $T'=T-\delta S$ and set $S'(c_1,\cdots,c_{n-1})=T'(c_1,\cdots,c_{n-1},e)-2T'(c_1,\cdots,c_{n-1},e)e$. Using the fact that T' is a cocycle (i.e.,$\delta T'=0$) we see that $\delta S'(c_1,\cdots,c_n)=\delta T'(c_1,\cdots,c_n,e)+(-1)^{n+1}T'(c_1,\cdots,c_{n-1},c_n)+(-1)^nT'(c_1,\cdots,c_n)e+(-1)^{n+1}\cdot T'(c_1,\cdots,c_{n-1},e)c_n$

It follows that the element $S+(-1)^{n+1}S'$ satisfies the requirement of the Lemma for $m=n$.

Lemma 3.2 If A, B, X is as in Lemma 3.1, and we put $X'=\{x\in X|xb=0=bx$ for all $b\in B\}$ then $H^n(A\oplus B, X)\cong H^n(A, X')\oplus H^n(B, X)$.

Proof Denote by I the identity operator on X, if T is an element of $L^n(A\oplus B, X)$, we define, for $a_1,\cdots,a_n\in A$, and $b_1,\cdots,b_n\in B$.

$$T_1(a_1,\cdots,a_n)=(I-e)T(a_1,\cdots,a_n)(I-e)$$

and $$T_2(b_1,\cdots,b_n)=eT(b_1,\cdots,b_n)e$$

It is easily verified that these mappings satisfy $(\delta T)_1=\delta T_1$ and $(\delta T)_2=\delta T_2$:

$$\begin{aligned}\delta T_1(a_1,\cdots,a_{n+1})&=a_1T_1(a_2,\cdots,a_{n+1})-T_1(a_1a_2,\cdots,a_{n+1})+\cdots\\&+(-1)^nT_1(a_1,\cdots,a_{n-1},a_na_{n+1})+(-1)^{n+1}T_1(a_1,\cdots,a_n)a_{n+1}\\&=(I-e)(\delta)(\delta T)(a_1,\cdots,a_{n+1})(I-e)=(\delta T)_1\end{aligned}$$

Similarly we have $(\delta T)_2=\delta T_2$. It follows that the mapping $T\to T_1+T_2$ induces a homomorphism of $H^n(A\oplus B, X)$ into $H^n(A, X')\oplus H^n(B, eXe)$.

Furthermore, if $T_1'\in Z^n(A, X')$ and $T_2'\in Z^n(B, eXe)$ and we define $T(a_1+b_1,\cdots,a_n+b_n)=T_1'(a_1,\cdots,a_n)+T_2'(b_1,\cdots,b_n)$ then $T\in Z^n(A\oplus B, X)$, $T_1=T_1'$(since $T_1'(a_1,\cdots,a_n)\in X'$) and $T_2=T_2'$(since $T_2'(b_1,\cdots,b_n)\in eXe$). Hence our mapping is an onto homomorphism.

Now consider a $T\in Z^n(A\oplus B, X)$ such that T_1 and T_2 are coboundaries. Then there exists $S_1'\in L^{n-1}(A, X')$ and $S_2'\in L^{n-1}(B, eXe)$ such that $T_1=\delta S_1'$ and $T_2=\delta S_2'$. On the other hand, applying Lemma 3.1 to T we obtain an $S\in L^{n-1}(A\oplus B, X)$

such that $(T-\delta S)(c_1 \cdots, c_n)=0$ Whenever one of the element c_i is equal to e. It is easily seen that this implies that:

$$(T-\delta S)(a_1+b_1, \cdots, a_n+b_n)=(T-\delta S)_1(a_1, \cdots, a_n)+(T-\delta S)_2(b_1, \cdots, b_n)$$

That is

$$(T-\delta S)(a_1+b_1, \cdots, a_n+b_n)=(\delta S_1'-\delta S_1)(a_1, \cdots, a_n)+(\delta S_2'-\delta S_2)(b_1, \cdots, b_n)$$

Now define

$$S'^*(a_1+b_1, \cdots, a_{n-1}+b_{n-1})=S_1'(a_1, \cdots, a_{n-1})+S_2'(b_1, \cdots, b_{n-1})$$

and $\quad S^*(a_1+b_1, \cdots, a_{n-1}+b_{n-1})=S_1(a_1, \cdots, a_{n-1})+S_2(b_1, \cdots, b_{n-1})$

Then

$$T-\delta S=(\delta(S'^*-S^*))_1+(\delta(S'^*-S^*))_2=\delta(S'^*-S^*) \text{ or } T=\delta(S-S^*+S'^*)$$

Thus our homomorphism is an isomorphism.

Finally by (b), $H^n(B, eXe) \cong H^n(B, X)$. We have completed the proof of lemma 3.2.

Corollary 3.3 Let A and B be Banach algebras over complex number field C. if X is a Banach $(A \oplus B)$-module, then $H^n(A \oplus B, X) \cong H^n(A, X) \oplus H^n(B, X)$.

Proof If A has an identity element e_1, We have $e_1 X' e_1 = e_1 X e_1$, where $H^n(A, X') \cong H^n(A, X)$. By (a) $H^n(\widetilde{A}, \widetilde{X}) \cong H^n(A, X)$ for all n. Thus we have $H^n(A \oplus B, X) \cong H^n(\widetilde{A \oplus B}, X) \cong H^n(\widetilde{A}, \widetilde{X}) \oplus H^n(\widetilde{B}, \widetilde{X}) \cong H^n(A, X) \oplus H^n(B, X)$.

Proposition 3.4 Let A be a Banach algebra. J is a complementable closed two-sided ideal of A. J and A/J are n-amenable algebras. then A is also n-amenable.

Proof By our hypothesis, we have $A \cong J \oplus A|_J$ Thus $H^n(A, X^*) \cong H^n(A/J, X^*) \oplus H^n(J, X^*)=0$.

When $n=1$, we obtain the result of ([3] proposition 5.1).

Lemma 3.5 Let A be a Banach algebra X a neo-unital A-module, J a closed two-sided ideal of A with a bounded approximate identity. Then for any $T \in Z^n(A, X^*)$, $n \in N$, and $S \in L^{n-1}(J, X^*)$ such that $\delta S = T|_J$, there is a $\widetilde{S} \in L^{n-1}(A, X^*)$ with $\delta \widetilde{S} = T$, $\widetilde{S}|_J = S$.

Proof We show by induction that for $i=0,1,\cdots, n-1$, there is $S_i \in L^{n-1}(A, X^*)$ with (a) $\delta S_i(a_1, \cdots, a_n)=T(a_1, \cdots, a_n)$, $a_1, \cdots, a_{n-i} \in J$, $a_{n-i+1}, \cdots, a_n \in A$ (b) $S_i|_J = S$.

Defineing $S_a(a_1, \cdots, a_{n-1})=S(a_1 e_a, \cdots, a_{n-1} e_a)$. We see that S_a is a bounded net in $L^{n-1}(A, X^*)$ and so has subset S_β convergent to a limit S_0 in the W^*-topology induced by identifying $L^{n-1}(A, X^*)$ with $L_{n-1}(A, X^*)$ (see [3]. p5) thus

$$\lim(X, S(a_1 e_\beta, \cdots, a_{n-1} e_\beta))=(X, S_0(a_1, \cdots, a_{n-1}))$$

for all $a_1, \cdots, a_{n-1} \in A$, $x \in X$.

Since $S(a_1 e_\beta, \cdots, a_{n-1} e_\beta) \to S(a_1, \cdots, a_{n-1}))$ in norm Thus $S_0|J=S$. Also $\delta S_0|_J = \delta S = T|_J$ So that S_0 satifies (a) and (b).

If S_i satisfies (a) and (b) put $T' = T - \delta S_i$ and define

$$S'_a(a_1, \cdots, a_{n-1}) = T'(a_1 e_a, \cdots, a_{n-i-2} e_a, e_a, a_{n-i-1}, \cdots, a_{n-1}).$$

As before we find a subsequence S'_v, W^*-convergent to $S' \in L^{n-1}(A, X^*)$. Note that $S'(b_1, \cdots, b_{n-1}) = 0$ for $b_1, \cdots, b_{n-1} \in J$. Since $T'(b_1 e_a, \cdots, b_{n-i-2} e_a, e_a, b_{n-i-1}, \cdots, b_{n-1})) - T'(b_1, \cdots, b_{n-i-2}, e_a b_{n-i-1}, \cdots, b_{n-1}) \to 0$ in norm as runs along the net we see that

$$\begin{aligned}
&(X, S'(b_1, \cdots, b_{n-i-2}, b_{n-i-1}, \cdots, b_{n-1})\\
&= \lim(X, T'(b_1 e_v, \cdots, b_{n-i-2} e_v, e_v, b_{n-i-1}, \cdots, b_{n-1}))\\
&= \lim(X, T'(b_1, \cdots, b_{n-i-2}, e_v, b_{n-i-1}, b_{n-1}))
\end{aligned}$$

Thus

$$\begin{aligned}
&(X, \delta S'(a_1, \cdots, a_{n-i-1}, a_{n-i}, \cdots, a_{n-1})\\
&= \lim(X, a_1 T'(a_2, \cdots, a_{n-i-1}, e_a, a_{n-i}, \cdots, a_n))\\
&\quad + \sum_{j=1}^{n-i-2} (-1)^j T'(a_1, \cdots, a_j a_{j+1}, \cdots, e_a, a_{n-i}, \cdots, a_n))
\end{aligned}$$

Where $a_1, \cdots, a_{n-i-1} \in J$. $a_{n-i-1}, \cdots, a_n \in A$.

And the remaining terms in $\delta S'$ are zero by (a):

$$\begin{aligned}
&(-1)^{n-i-1} S'(a_1, \cdots, a_{n-i-1} a_{n-i}, a_{n-i+1}, \cdots, a_n)\\
&+ \sum (-1)^j S'(a_1, \cdots, a_{n-i-1}, a_{n-i}, \cdots, a_j a_{j+1}, \cdots, a_n)\\
&= \lim((-1)^{n-j-1} T'(a_1, \cdots, a_{n-i-2}, e_a, a_{n-i-1} a_{n-i}, a_{n-i+1}, \cdots, a_n)\\
&+ \sum (-1)^j T'(a_1, \cdots, a_{n-i-2}, e_a, a_{n-i-1}, a_{n-i}, \cdots, a_j a_{j+1}, \cdots, a_n) = 0
\end{aligned}$$

Comparing this with the equatton

$$\delta T'(a_1, \cdots, a_{n-i-1}, e_a, a_{n-i}, \cdots, a_n) = 0$$

we see that

$$\begin{aligned}
&(X, \delta S'(a_1, \cdots, a_{n-i-1}, a_{n-i}, \cdots, a_n)\\
&= \lim_a(X, (-1)^{n-i} T'(a_1, \cdots, a_{n-i-2}, a_{n-i-1}, e_a, a_{n-i}, \cdots, a_n)\\
&= (X, (-1)^{n-i} T'(a_1, \cdots, a_{n-i-1}, a_{n-i}, \cdots, a_n)
\end{aligned}$$

Thus $S_{i+1} = S_i + (-1)^{n-i} S'$ has the required properties. The Lemma then follows by taking $S = S_{n-1}$ because

$$(T - \delta\widetilde{S})(a_1, \cdots, a_n) = 0 \text{ if } a_1 \in J$$

and as $\delta(T - \delta\widetilde{S})(a_1, \cdots, a_{n+1}) = 0$ taking $a_1 \in J$

we get $a_1(T - \delta\widetilde{S})(a_2, \cdots, a_{n+1}) = 0$

Now if $y \in X^*$ and $a_1 y = 0$ for all $a_1 \in J$ then $0 = (x, a_1 y) = (x a_1, y)$ for all $x \in X$, $a_1 \in J$. So that $y = 0$ as X is neo-unital we thus see $T = \delta\widetilde{S}$.

Proposition 3.6 Let A be a Banach algebra, J be a closed two-sided ideal of A with a bounded approximate identity, and J be n-amenable. then A is also n-amenable.

Proof By ([3] proposition 1.8), it is enough to show that $H^n(A, X^*) = 0$ for any neo-unital A-module X. Given $T \in Z^n(A, X^*)$, then there is an $S \in L^{n-1}(J, X^*)$ such $\delta S = T|_J = S$, since J is n-amenable. So we have, by Lemma 3.5,

$\widetilde{S}\in L^{n-1}(A, X^*)$. With $\delta\widetilde{S}=T, \widetilde{S}|_J=S$. Thus $H^n(A, X^*)=0$.

Moreover, for any Banach algebra A, let $\widetilde{A}$ be A if A has a unit and be the unitarization of A otherwise. Then $\widetilde{A}$ is n-amenable if A is n-amenable.

Proposition 3.7 Let A be a 2-amenable algebra, J be a closed two-sided ideal of A. Then the quotient algebra A/J is also 2-amenable.

Proof Suppose X is an A/J-module, Hence an A-module, and $Q^{(2)}: A\times A \to A/J\times A/J$ is the produt of the classical quotient of A into A/J. For any $T\in Z^2(A/J, X^*)$, $TQ^{(2)}\in Z^2(A, X^*)$. Thus there is an $S\in L^1(A, X^*)$ Such that $TQ^{(2)}=\delta S$ and $TQ^{(2)}|_J=0$, hence $\delta S(b_1, b_2)=b_1S(b_2)-S(b_1b_2)+S(b_1)b_2=0$, $b_1, b_2\in J$, so $S|_J\in Z^1(J, X^*)=B^1(J, X^*)$. This implies that $S|_J=\delta y$ for some $y\in X^*$, and $\delta y(a)=ay-ya$, but the action of J on X^* is trivial, So $S|_J=0$, There is an $\widetilde{S}: A/J\to X^*$ Such that $S=\widetilde{S}Q^{(2)}$ Thus $T=\delta\widetilde{S}$.

Proposition 3.8 If A is 2-amenable, and J is a closed two-sided ideal with an approximate identity, then J is 2-amenable.

Proof It is enough to show $H^2(J, X^*)=0$ for all unital J-module X ([3] propositionl.8). By ([14].proposition 3.7) there is an algebraic homomorphism $Q: A\to \Delta(J)$ such that X becomes an A-module. By ([3] proposition .9), for any $D\in Z^2(J, X^*)$ there is a $\widetilde{D}\in Z^2(\Delta(J), X^*)$, such that $\widetilde{D}|_J=D$. By the proof of ([3] proposition 1.8) $\widetilde{D}Q^{(2)}\in Z^2(A, X^*)$, hence $DQ^{(2)}=\delta^2T$ for some $T\in L^1(A, X^*)$, so $D=\delta(T|_J)$ since the restriction of Q to J is the identity map.

Let A be a Banach algebra, J a closed two-sided ideal of A with an approximate identity. By proposition 3.6 and prposition 3.8, A is 2-amenable if and only if J is 2-amenable. If A is a C^*-algebra then every (two-sided) closed ideal of A has a bounded approximate identity ([2] Lemma 4.2.11). Hence A is 2-amenable if and only if one of the two-sided ideals of A is 2-amenable.

Proposition 3.9 Let A, B be two Banach algebras, and A is 2-amenable with an approximate, then $A\widehat{\otimes}B$ is 2-amenable.

Proof It is trivial by proposition 3.6.

4. Criterion of amenable algebras

Theorem 4.1 Any Banach algebra A with an approximate unit is left 2-amneable.

Proof Let A be a Banach algebra with an approximate identity, and N be a w^*-adherent point of $(e_a\otimes e_a)$ in $(A\widehat{\otimes}A)^{**}$. Then $Na=\lim(e_a\otimes e_a)a=\lim(e_ab\otimes e_ac)=b\otimes c\in A\widehat{\otimes}A$ and $\pi(Na)=\pi(\lim(e_a\otimes e_a)a)=\pi(b\otimes c)=bc=a$. Where $a=bc$, by factorization theorem.

Assum that T is a continuous bilinear map from A into X^* with $a^3T=0$ while

X satisfies $ax=0$ for all $a\in A$, $x\in X$. Define

$T': A\hat{\otimes} A\to X^*$ by $T'(b\otimes c)=T(b,c)$ and $S\in L^1(A, X^*)$, $S(a)=T'(Na)$, Then

$$\delta^2 S(a,b)=a(b)-S(ab)+S(a)b=-S(ab)+S(a)b=T'(Nab)-T'(Na)b$$
$$=T'(Na_1a_2b)-T'(Na_1a_2)b \qquad (a=a_1a_2)$$
$$=T'(a_1\otimes a_2b)-T'(a_1\otimes a_2)b=T(a_1,a_2b)-T(a_1,a_2)b$$
$$=T(a,b)$$

Then $T=\delta^2 S$.

Theorem 4.2 Any commutative Banach algebra with an approximate identity is 2-amenable.

Proof Suppose A is a commutative Banach algebra with an apporoximate identity, and $\widetilde{A}$ is the unitarization of A. we will show that $H^2(A, X^*)=0$, for any neo-unital A-module. Given $T\in Z^2(\widetilde{A}, X^*)$, then

$$0=\delta T(a,b,c)=aT(b,c)-T(ab,c)+T(a,bc)-T(a,b)c$$

Take $c=e$, then

$$aT(b,e)=T(ab,e), \text{ Take } a=e, \text{ then } T(e,bc)=T(e,b)c.$$

Setting $S(a)=T(a,e)=aT(e,e)$, then we have

$$(x,\delta S(a,b))=(x, aS(b)-S(ab)+S(a)b)$$
$$=(x, aS(e)b-aS(e)b+aS(e)b)=(bxa, S(e))$$
$$=(x, aT(e,e)b)=(x, T(a,b))$$

for all $a,b\in A$. Hence $T(a,b)=\delta S(a,b)$.

Corollary 4.3 All commutative C^*-algebras are 2-amenable.

Proof Every C^*-algebra has an approximate identity, so every commutative C^*-algebra is 2-amenable, by Theorem 4.2.

References

[1] Waiter Rubin, Functional Analysis.

[2] Richard V, kadison John R Ringrose Fundmentals of the theory of operator Algebras ACADE MEIC PRESS, INC 1983.

[3] B. E. Johnson, Cohomology in Banach Aigebras, Memoirs of the Amerocam Math. Soc. 127 (1972).

[4] B. E. Johnson, An intorduction to the theory of centralizars. Prp. London (3) 14 (1964) 299—320.

[5] B. E. Johnson, Approximate diagonals and cohomology of certain annihilatror Banach algebras, Amer. J. Math. 94 (1972) 685—698.

[6] B. E. Johnson, Low dimensional cohmology of Banach algebras. Proc. of symposia in pure [illegible]. Vol 38 1982 [illegible] 2.

[7] J.W. Bunnce, Characterizations of amenbable and strongly amneable C^*-algebra, Pacific J. Math. 43(1972) 563—572. MR47.

[8] J. W. Bunce, Finite operators and amenable C^*-algebras, Proc. American Math.Soc. 56(1976) 145—151.

[9] J. Dixmier, Les C^*-algebres et leure representations cahires socentifiques fasc, 29 Gauthier-viclars Paris, 1964 Mr 30 $^\#$.

[10] G. Hochschild, On the cohomology groups of associative algebra, Ann. of Math. 46(1945) 58—67.

[11] G. Hochschild, On the cohomology theory for associative algebras, Ann. of Math. 47(1946) 568—579.

[12] J.R. Ringrose, Cohomology of Banach algebras, Lectures on operator algebras 359—434 (springer Lecturés Notes in Math. No. 247).

[13] R. V. Kadison and J. R. Ringrose, Cohomology of operator algebras Ⅱ Extended coboundaries and the hyperfinite case. Ark Mat. 9(1971) 55—63.

[14] Robert C. Busby, Double centralizars and extension of C^*-algebras.

[15] Josph J. Rotman, An introduction to homological algebras, Academic press, New York, San Francisco, London 1979.

[16] A. Ja. Helemskiy, On the homological demension of normed modules over Banach algebras, (in Russian), Mat. skornik Vol. 81(123)(9170) pp430—444(English translation) Math. USSR. sbornik Vol.10(1970) p399—411.

[17] Jaan-paul Pier, Amenable locally compact groups, New York, Chichester, Brisbane, Toronto, singapore 1984.

[18] 邓宏钧、高仁、张政修。董伦群，算子代数上同调理论，武汉出版社，1988.

[19] 李炳仁，算子代数，科学出版社，1986.

[20] 周伯埙，同调代数，科学出版社，1988.

[21] Deng Hongjan, Hu Yaohua, On Amenable Algebras, The Asian Mathematical Conference, 1990 Hong Kong, 14—18 Angust 1990.