Journal of Mathematical Research and Exposition
Vol.11 No.4 Nov., 1991

Results on Commutativity of Rings*

H. A. S. Abujabal *M. S. Khan*

(Dept. Math. Faculty of Science, King Abdul Aziz University
P. O. Box 31464, Jeddah-21497, Suadi Arabia)

Abstract

We establish two theorems on the commutativity of rings with identity. While our first result is based on an iteration method, the second deals with central conditions.

1. Introduction

Many sufficient conditions are wellknown under which a given ring becomes commutative. Recently, Psomopoulos [3] and Harmanci [2] proved some interesting theorems on the commutativity of rings. Our objective in this note is to present two results concerning ring commutativity which are related to the work of earlier authors.

Throughout the rest of the paper, R stands for a ring and $Z(R)$ denotes its centre. As usual, $[x, y] = xy - yx$. Following results will be frequently used in the sequel.

Lemma 1.1 (Tong [5]) Let R be a ring with unity 1. Let $I_0^r(x) = x^r$. If $k >$ 1, let $I_k^r(x) = I_{k-1}^r(1+x) - I_{k-1}^r(x)$. Then $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!x$; $I_r^r(x) = r!$, $I_j^r(x) = 0$, for $j > r$.

Theorem 1.2 (Harmanci [2]), p. 28) If for every x and y in R (not necessarily with a unity) we can find a polynomial $p_{x,y}(t)$ with integer coefficients which depend on x and y such that $[x^2 p_{x,y}(x) - x, y] = 0$, then R is commutative.

Lemma 1.3 (Quadri et al. [4]). Let R be a prime ring and x be a nonzero central element such that x, y is central for some y in R. Then y must also be central.

2. Results

A generalization of a famous result due to Bell [1] was obtained by Psomopoulos [3] through a tedious and lengthy proof. Here we present an entirely

* Received July 19, 1990.

different but easier and shorer proof of a speeial case of the main theorem of Psomopoulos [3]. Following the lines of proof of Harmanci [2], a new but similar result will be obtained.

Theorem 2.1 Let R be a ring with unity 1. Let $n(>1)$ and m be two fixed positive integers, and t be any non-negative integer. Suppose that the polynomial identity

$$(*) \qquad x^t[x^n, y] = [x, y^m]y$$

holds for all x, y in R. Then R must be commutative provided that every commutator in it is $m(m!)$-torsion free.

Proof As in Lemma 1.1, let $I_j(y) = I_j^m(y)$, for $j = 0, 1, 2, \cdots$. Then the polynomial identily $(*)$ of the theorem can be re-writen as

$$(1) \qquad x^t[x^n, y] = [x, I_0(y)]y, \text{ for all } x, y \text{ in } R.$$

Replace y by $y+1$ in (1) to obtain

$$x^t[x^n, y] = [x, I_0(y+1)](y+1), \text{ for all } x, y \text{ in } R.$$

By Lemma 1.1, the above relation for all x, y in R gives

$$x^t[x^n, y] = [x, I_0(y)]y + [x, I_1(y)]y + [x, I_0(y)] + [x, I_1(y)].$$

Combining the above expression with (1), we obtain

$$(2) \qquad [x, I_1(y)]y + [x, I_0(y)] + [x, I_1(y)] = 0,$$

ior all x, y in R.

Now, let $y = y+1$ in (2) we have for all x, y

$$[x, I_1(y+1)](y+1) + [x, I_0(y+1)] + [x, I_1(y+1)] = 0.$$

Then again by Lemma 1.1 and (2), we get

$$(3) \qquad [x, I_1(y)] + 2[x, I_2(y)] + 2[x, I_1(y)] = 0,$$

for all x, y in R.

Once again replace y by $y+1$ in the above expression and then applying the previous arguments, we have

$$(4) \qquad [x, I_3(y)]y + 3[x, I_3(y)] + 3[x, I_2(y)] = 0,$$

for all x, y in R.

Finally, after replacing y by $y+1$ in (4) and then iterating m times, we obtain for all x, y in R, the relation

$$(5) \qquad [x, I_m(y)]y + m[x, I_m(y)] + m[x, I_{m-1}(y)] = 0.$$

But $I_{m-1}(y) = \frac{1}{2}(m-1)m! + m!y$ and $I_m(y) = m!$, by Lemma 1.1. So the equation (5) is reduced to $mm!\,[x, y] = 0$, for all x, y in R. Clearly, R is commutative as every commutator in it is $mm!$-torsion free. This completes the proof.

We omit the proof of the following theorem since it is similar to Theorem 2.1.

Theorem 2.2 Let R be a ring with unity. Let $n(>1)$ and m be two fixed positive integers, and s be any non-negative integer. Suppose that the polynomial identity

(**) $$x[x^n, y]=[x, y^m]y^s$$

holds for all x, y in R. Then R must be commutative provided that every commutator in it is $nn!$-torsion free.

Remark It will be of interest to develope an iteration technique so as to deal with conditions such as

(***) $$x^t[x^n, y]=[x, y^m]y^s,$$

which was considered by Psomopoulos [3]. Here m, n, t and s are as in Theorem 2.1 and Theorem 2.2.

In order to obtain Harmanci type commutativity theorem, we first establish the following.

Lemma 2.3 Let R be a prime ring with unity 1 satisfying

(i) $[x^n, y]-[x, y^{n+1}]\in z(R)$, (ii) $[x^{n+1}, y]-[x, y^{n+2}]\in z(R)$,

for all x, y in R and a fixed integer $n>1$. Then R is commutative.

Proof Replace x by $(1+x)$ in (i) we get $[(1+x)^n, y]-[x, y^{n+1}]\in z(R)$. But we can write

$$[(1+x)^n, y]=n[x, y]+\sum_{k=2}^{n-1}\binom{n}{k}[x^k, y]=[x^n, y].$$

Combining the above two expressions we obtain

(a) $$n[x, y]+\sum_{k=2}^{n-1}\binom{n}{k}[x^k, y]\in z(R).$$

Similarly, from (ii) we can obtain

(b) $$(n+1)[x, y]+\sum_{j=0}^{n}\binom{n+1}{j}[x^j, y]\in z(R)$$

Now, substracting (b) from (a), we have

$$\left[\sum_{k=2}^{n-1}\binom{n}{k}x^k-\sum_{j=2}^{n}\binom{n+1}{j}x^j, y\right]-[x, y]\in z(R).$$

whence we get

(c) $$[x^2p(x)-x, y]\in z(R) \text{ for all } x, y \text{ in } R.$$

Here $p(x)$ is a polynomial with integer coefficients and may depend on x and y. Now replacing y by yx in (c), we see that

(d) $$[x^2p(x)-x, y]x\in z(R), \text{ for all } x, y \text{ in } R.$$

As R is prime, using (c), (d) and Lemma 1.3, we find that $x\in z(R)$ unless $[x^2p(x)-x, y]=0$ for all x, y in R. Then by Theorem 1.2, R must be commutative.

Theorem Let R be a semi prime ring with unity 1 satisfying all the conditions of Lemma 2.3. Then R is commutative.

Proof Being semiprime, R is isomorphic to a subdireot sum of prime rings R_a, each of which as homomorhic image of R satisfies the hypothesis placed on R. By Lemma 2.3, each of R_a is commutative. Hence R is commutative. This ends the proof.

References

[1] H. E. Bell, On the power map and ring commutativity, Canad. Math. Bull., 21(1978), 399—404.

[2] A. Harmanci, Two elementary commutativity theorems for rings, Acta Math. Acad. Sci. Hunger., 29(1977), 23—29.

[3] E. Psomopoulos, Commutativity theorems for rings and groups constraints on commutators, Internat. J. Math. & Math. Sci., 7(3)(1984), 513—517.

[4] M. A. Quadri, M. A. Khan and M. Ashraf, A commutativity theorem for semi-prime rings, Math. Student, 50(1982), 264—267.

[5] J. Tong, On the commutativity of a ring with identity, Canad. Math. Bull., 27(4)(1984), 456—460.