Journal of Mathematical Research and Exposition
Vol. 11, No.4 Nov., 1991

On Hyperspace Topologies of Some Quasiuniform Spaces*

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Abstract

In this paper, it is shown that the locally finite topology e^{τ} on the hyperspace 2^X coincides with the topology transmitted by the locally finite covering quasiuniformity on X. We also prove that the following conditions are equivalent: (1) (X,τ) is paracompact, (2) (X,τ) is orthcompact, and $e^{\tau}=|2^{u_{FT}}|$, (3) $e^{\tau}=|2^{u_L}|$ for some Lebesgue quasiuniformity u_L. A characterization of feebly compact topological spaces is given.

§ 1 Introduction

In recent years, the locally finite topology on the hyperspace 2^X was discussed by some authors. For example, Beer et al [1] proved that the locally finite topology is the supremum of all the Hausdorff metric topologies corresponding to equivalent metrics for a metrizable space X, Naimpally and Sharma[7] proved that, for a normal, the locally finite topology coincides with the topology transmitted by the fine uniformity on X. Since the results of [7] are not valid in the case of quasiuniformities, we will consider some other special quasiuniformities in this paper, e.g. the locally finite covering quasiuniformity, and Lebesgue quasiuniformities. We prove that the locally finite topology on 2^X coincides with the topology transmitted by the locally finite covering qugsiuniformity and that the locally finite topology is weaker than the topology transmitted by a Lebesgue quasiuniformity. From these, the characterizations of feebly compactness and paracompactness of a topological space follow.

§ 2 Preliminaries

Let $(X,\mathscr{U})$ be a quasiuniform space (all axioms for a uniform space hold except for perhaps the symmetry axiom). $|\mathscr{U}|$ denotes the topology induced by u. $\mathscr{U}^{-1}=\{U^{-1}: U\in\mathscr{U}$ is also a quasiuniformity on X which is called the conjugate quasiuniformity of $\mathscr{U}$. For each $U\in\mathscr{U}$, as in [5], set

$$H(U)=\{(A,B): A,B\in 2^X,\ B\subset U(A),\ A\subset U((B)\}.$$

Then $\{H(U): U\in\mathscr{U}$ is a base for a quasiuniformity 2^u on 2^X.

* Received Mar. 19, 1990.

Definition 2.1 A quasiuniform space $(X,\mathscr{U})$ is equinormal if for disjoint closed subsets A,B there is a $U\in\mathscr{U}$ such that $A\times B\cap U=\phi$.

It is easy to show that (X,u) is equinormal if and only if for each open neighborhood G of a closed set F with $F\subset G$, there is a $U\in\mathscr{U}$ such that $F\subset U(F)\subset G$.

Definition 2.2 A quasiuniform space $(X,\mathscr{U})$ is precompact provided that for each $U\in\mathscr{U}$, there is a finite subset F of X such that $X=U(F)$.

Definition 2.3 A quasiuniform space $(X,\mathscr{U})$ is totally bounded provided that for each $U\in\mathscr{U}$ there is a finite cover $\mathscr{A}$ of X such that for each $A\in\mathscr{A}$, $A\times A\subset U$.

It is well known that $(X,\mathscr{U})$ is totally bounded iff both $(X,\mathscr{U}$ and $(X,\mathscr{U}^{-1})$ are precompact. And totally boundedness is a hereditary property.

Let (X,τ) be a topological space. A Q-cover of X is an open cover $\mathscr{C}$ such that $\mathscr{C}(X)\in\tau$ for each $x\in X$, where $\mathscr{C}(X)=\bigcap\{C: x\in C\in\mathscr{C}\}$. X is orthcompact if every open cover of X has a Q-cover refinement. Let γ be a collection of Q-covers of X. For each $\mathscr{C}\in\gamma$, set

$$U_{\mathscr{C}}=\bigcup_{x\in X}\{x\}\times\mathscr{C}(x) \quad \text{and} \quad \varphi=\{U_{\mathscr{C}}:\mathscr{C}\in\gamma\}.$$

It is known that if γ is the collection of all Q-covers, locally finite open covers, finite open covers, then φ is a subbase for the fine transitive, locally finite covering and Pervin quasiuniformity for $(X,\mathscr{U})$, respectively. We denote by $\mathscr{U}_{FT}$, $\mathscr{U}_{LF}$, and $\mathscr{U}_p$, respectively, and let $\mathscr{U}_F(\mathscr{U}_F^*)$ denote the fine (totally bounded) quasiuniformity for (X,τ).

We say that a quasiuniformity $\mathscr{U}$ is compatible with a topological space (X,τ) iff $\tau=|\mathscr{U}|$. It is easy to see that $\mathscr{U}_{FT}$, $\mathscr{U}_{LF}$, $\mathscr{U}_p$, $\mathscr{U}_F$, $\mathscr{U}_F^*$ on (X,τ) are all compatible quasiuniformities.

The hyperspace 2^X of (X,τ) is the set $\{E\subset X: E \text{ is closed}, E\neq\phi\}$. For a collection $\mathscr{A}$ of subsets of X, set

$$\mathscr{A}^-=\{E\in 2^X: E\cap A\neq\phi \quad \forall A\in\mathscr{A}\},$$
$$\mathscr{A}^+=\{E\in 2^X: E\subset\bigcup\mathscr{A}\},$$
$$\mathscr{A}^\sim=\mathscr{A}^+\cap\mathscr{A}^-,\ \mathscr{A}(x)=\bigcap\{A: x\in A\in\mathscr{A}\}.$$

The Vietoris (locally finite) topology $2^\tau(e^\tau)$ on 2^X is the one generated by the sets of the form $\mathscr{A}^\sim$, where $\mathscr{A}$ is any finite (locally finite) collection of open subsets of X.

All topological spaces considered in this paper are assumed to be T_1. Basic concepts and terminology about quasiuniform spaces and hyperspaces are referred to [4] and [6].

§ 3 Main Results

In this section, we will discuss the locally finite topology and topologies transmitted by some covering-type quasiuniformities.

Theorem 3.1 Let $(X,\mathscr{U})$ be a quasiuniform space.

(1) $|2^{u}|\subset 2^{|u|}$ if and only if (X,u^{-1}) is hereditarily precompact to closed subsets.

(2) $2^{|u|}\subset|2^{u}|$ if and only if $(X,\mathscr{U})$ is equinormal.

Proof (1) If $|2^{u}|\subset 2^{|u|}$, then for every $A\in 2^{X}$ and every $U\in\mathscr{U}$, there is a finite collection $\mathscr{C}=\{0_1,0_2,\cdots,0_n\}$ of open subsets such that $A\in\mathscr{C}^{\wedge}\subset H(U)(A)$. Pick $x_i\in A\cap 0_i$ $(1\leqslant i\leqslant n)$. Then $A\subset\bigcup_{i=1}^{n}U^{-1}(x_i)$. Therefore $(X,\mathscr{U}^{-1})$ is hereditarily precompact to $A\in 2^{X}$. Conversely, for each $U\in\mathscr{U}$, and $A\in 2^{X}$, take $V\in\mathscr{U}$ with $V\circ V\subset U$, there is $\{x_1,x_2,\cdots,x_n\}\subset A$ such that $A\subset\bigcup_{i=1}^{n}V^{-1}(x_i)$. It is easy to show that $A\in(\mathrm{int}V(A))^{+}\cap\{\mathrm{int}V(x_1),\cdots,\mathrm{int}V(x_n)\}^{-}\subset H(U)(A)$. Therefore $|2^{u}|\subset 2^{|u|}$.

(2) If $2^{|u|}\subset|2^{u}|$, then for each open neighborhhod G of $A\in 2^{X}$, there is a $U\in\mathscr{U}$ such that $H(U)(A)\subset\{G\}^{+}$. It is easy to see $U(A)\in H(U)(A)\subset\{G\}^{+}$. So $(X,\mathscr{U})$ is equinormal. Conversely let $\mathscr{C}=\{0_1,0_2,\cdots,0_n\}$ be a finite collection of open subgets of $(x,\mathscr{U})$, $A\in 2^{X}$ with $A\in\mathscr{C}^{\wedge}$. Take $x_i\in A\cap 0_i$ and $U_i\in\mathscr{U}$ such that $U_i(x_i)\subset 0_i$. Take $U_0\in\mathscr{U}$ such that $A\subset U_0(A)\subset\bigcup_{i=1}^{n}0_i$. Set $U=(\bigcap_{i=1}^{n}U_i)\cap U_0$. It follows that $A\in H(U)(A)\subset\mathscr{C}^{\sim}$.

As special cases, we have

Corollary 3.2[5] $|2^{u_P}|=2^{\tau}$ for any topological space (X,τ).

Corollary 3.3 $|2^{u_{\dot{F}}}|=2^{\tau}$ for any topological space (X,τ).

Theorem 3.4 Let (X,τ) be a topological space. Then $e^{\tau}=|2^{u_{LF}}|$.

Proof Let $\mathscr{A}$ be a locally finite collection of open subsets and $F\in\mathscr{A}^{\sim}\cap 2^{X}$. Then $\mathscr{B}=\mathscr{A}\cup\{X-F\}$ is a locally finite open cover of X. Then $F\in H(U_{\mathscr{B}})(F)\subset\mathscr{A}^{\sim}$. In fact, if $G\in H(U_{\mathscr{B}})(F)$, then $G\subset U_{\mathscr{B}}(F)=\bigcup_{x\in F}\mathscr{B}(x)\subset\bigcup\mathscr{A}$. For $\forall A\in\mathscr{A}$, take $x_0\in A\cap F$, then $x_0\in\mathscr{B}(x_0)\subset A$. Take $y_0\in G$ such that $y_0\in U_{\mathscr{B}}(x_0)\subset A$, *i.e.*, $y_0\in G\cap A$. So we have $e^{\tau}\subset|2^{u_{L}}|$.

On the other hand, let $\mathscr{A}_i$ $(1\leqslant i\leqslant n)$ be a locally finite open cover. By Theorem 1.42 of [5], we know that $H(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})$ is a member of a base for $2^{u_{LF}}$. For $F\in 2^{X}$, set $\mathscr{B}=\{\bigcap_{i=1}^{n}\mathscr{A}_i(x):x\in F\}$. Then $\mathscr{B}$ is a locally finite collection of open subsets of X. We have $F\in\mathscr{B}\subset H(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(F)$. In fact, if $G\in\mathscr{B}$, then $G\subset\bigcup\mathscr{B}=$

$\bigcup_{x\in F}(\bigcap_{i=1}^{n}\mathscr{A}_i(x))=(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(F)$. For $x_0\in F$, take $y_0\in G\cap(\bigcap_{i=1}^{n}\mathscr{A}_i(x_0))$, then $y_0\in(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(x_0)$, so $F\subset(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})^{-1}(G)$. Therefore $|2^{\mathscr{U}_{LF}}|\subset e^{\tau}$.

Corollary 3.5 $e^{\tau}\subset|2^{\mathscr{U}_F}|$, for any topological space (X,τ).

A topological space is feebly compact provided that each locally finite collection of open subsets is finite. The following theorem is a characterization of this kind of spaces.

Theorem 3.6 Let (X,τ) be a topological space. Then the following conditions are equivalent

(1) (X,τ) is feebly compact,

(2) $\mathscr{U}_{LF}=\mathscr{U}_P$,

(3) $(X,\mathscr{U}_{LF}^{-1})$ is hereditarily precompact to closed subsets.

(3) $|2^{\mathscr{U}_{LP}}|=2^{\tau}$,

(5) $e^{\tau}=2^{\tau}$.

Proof: (1)⇒(2) is trivial. (2)⇒(3) since $\mathscr{U}_p$ is totally bounded. Clearly (3)⇒(4) follows from Theorem 3.1. And the implication (4)⇒(5) follows from Theorem 3.4. (5)⇒(1) is from Theorem 1.2 of [7].

We say that a compatible quasiuniformity $\mathscr{U}$ on a topological space (X,τ) is a Lebesgue quasiuniformity provided that for each open cover $\mathscr{A}$ of X, there is an $U\in\mathscr{U}$ such that $\{U(x);x\in X\}$ refines $\mathscr{A}$. It is well known in [4] that (X,τ) is paracompact if and only if $\mathscr{U}_{LF}$ is a Lebesgue quasiuniformity, and (X,τ) is orthcompact if and only if $\mathscr{U}_{FT}$ is a Lebesgue quasiuniformity.

Theorem 3.7 Let $\mathscr{U}_L$ be a compatible Lebesgue quasiuniformity on a topological space (X,τ). Then $e^{\tau}\subset|2^{\mathscr{U}_L}|$.

Proof: Let $\mathscr{A}=\{A_i;i\in I\}$ be a locally finite collection of open subsets. For $F\in 2^X$, $F\in\mathscr{A}^{\wedge}=\mathscr{A}^{+}\cap\mathscr{A}^{-}$. For each $i\in I$, take $a_i\in A_i\cap F$, then the set $E=\{a_i;i\in I\}$ is diserete. For $e\in E$, set

$$V_e=\bigcap\{A_i;e\in A_i\}-(E-\{e\});$$

note that V_e is an open set with $V_e\cap E=\{e\}$. Then $\{V_e;e\in E\}\cup\{A_i-E;i\in I\}$ is a locally finite collection of open subsets which refines $\mathscr{A}$. Take $U\in\mathscr{U}$ such that $\{U(x);x\in X\}$ refines $\{V_e;e\in E\}\cup\{A_i-E;i\in I\}\cup\{X-F\}$. We claim that $F\in H(U)(F)\subset\mathscr{A}^{\sim}$. In fact, for each $G\in H(U)(F)$, since $\{U(x);x\in X\}$ also refines $\mathscr{A}\cup\{X-F\}$, $G\subset U(F)\subset\bigcup\mathscr{A}$, i.e., $G\in\mathscr{A}^{+}$. For each $i\in I$, there is $y_i\in G$ such that $y_i\in U(a_i)$. Since $U(a_i)$ can only lie in V_{a_i}, so that $y_i\in U(a_i)\subset V_{a_i}\subset A_i$, and hence we see that $G\in\mathscr{A}^{+}\cap\mathscr{A}^{-}=\mathscr{A}^{\sim}$. It follows that $e^{\tau}\subset|2^{\mathscr{U}_L}|$.

Now we characterize paracompactness in terms of Lebesgue quasiuniformities.

Theorem 3.8 Let (X,τ) be a topological space. Then the following conditions are equivalent

(1) (X,τ) is paracompact;

(2) (X,τ) is orthcompact, and $e^{\tau}=|2^{\mathscr{U}_{FT}}|$;

(3) $e^{\tau}=|2^{\mathscr{U}_L}|$ for some compatible Lebesgue quasiuniformity $\mathscr{U}_L$ on (X,τ).

Proof (1)⇒(2). It suffices to prove that $|2^{\mathscr{U}_{FT}}|\subset e^{\tau}$. Let $\mathscr{A}_i(1\leqslant i\leqslant n)$ be a Q-cover of X. Then $H(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})$ is a member of a base for $2^{\mathscr{U}_{FT}}$. We consider $A\in H(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(A)$ for $A\in 2^X$. Since (X,τ) is paracompact, then there is a locally finite open cover $\mathscr{B}$ which refines $\{\bigcap_{i=1}^{n}\mathscr{A}_i(x):x\in X\}$. Set $\mathscr{C}(x)=\mathscr{B}(x)\cap(\bigcap_{i=1}^{n}\mathscr{A}_i(x))$ for each $x\in A$ and $\mathscr{C}=\{\mathscr{C}(x):x\in A\}$. Then $\mathscr{C}$ is a locally finite collection of open subsets. We claim that $A\in\mathscr{C}^{\wedge}\subset H(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(A)$. In fact, for each $G\in\mathscr{C}^{\wedge}$, it is sasy to show $G\subset(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(A)$. For each $x_0\in A$, take $y_0\in G\cap\mathscr{C}(x_0)$ then $y_0\in(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(x_0)$. Therefore $G\in H(\bigcap_{i=1}^{n}U_{\mathscr{A}_i})(A)$. By Theorem 3.7, $e^{\tau}=|2^{\mathscr{U}_{FT}}|$ follows.

(2)⇒(3). It is trivial.

(3)⇒(1). Let $\mathscr{A}$ be an open cover of X. Then there is an $U\in\mathscr{U}_L$ such that $\{U(x):x\in X\}$ refines $\mathscr{A}$. Since $e^{\tau}=|2^{\mathscr{U}_L}|$, there is a locally finite open cover $\mathscr{B}$ of X such that $X\in\mathscr{B}^{\sim}\subset H(U)(X)$. Then $\{\mathscr{B}(x)\cap U(x):x\in X\}$ is a locally finite open cover of X which refines $\mathscr{A}$. Therefore (X,τ) must be paracompact.

Let (X,τ) be a orthcompact topological space (not paracompact). By above theorems, $|2^{\mathscr{U}_{FT}}|\not\subset e^{\tau}$. This implies that Theorem 2.1 (1) and Theorem 2.2 of [7] can not be generalized to quesiuniform spaces.

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关于某些拟一致空间的超空间拓扑

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摘　　要

设（X,τ）是一个拓扑空间。在本文中，我们证明了在超空间2^X上局部有限拓扑e^τ与局部有限覆盖拟一致$\mathcal{U}_{LF}$所导出的超拓扑$|2^{\mathcal{U}_{LF}}|$是相同的。我们还证明了下面条件是等价的：（1）（X,τ）是仿紧的；（2）（X,τ）是orth-紧的，且$e^\tau=|2^{\mathcal{U}_{FT}}|$；（3）存在一个Lebesgue拟一致$\mathcal{U}_L$，使$e^\tau=|2^{\mathcal{U}_L}|$。同时，我们也给出了feebly—紧拓扑空间的一个特征刻划。