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On the Enumeration of Labeled Graphs with k Cycles*

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Abstract

Enumeration of labeled connected graphs with k cycles is an open problem. Up to now it has been settled only for $k=1,2$. Unfortunately, these given formulas are very complicated. In this paper we improve a series of enumnerations and solve the enumeration problem for $k=3$.

I. Introduction

Let $C(n,k)$ denote the number of labeled connected graphs of order n with k cycles.

In 1959, Renyi [2] gave the formula for $k=1$ as follows:

Theorem A $C(n,1)=\frac{1}{2}\sum_{p=3}^{n}[n-1]_{p-1}n^{n-p}$,

where $[m]_r=m(m-1)\cdots(m-r+1)$, $(m\geq r>0)$.

In 1972, Tomescu [3] gave the following furmula for a spacial case.

Theorem B Let $C^{(1)}(n,k)$ be the number of labeled connected graphs of order n with k cycles which have no vertices in common each other, then

$$C^{(1)}(n,k)=\frac{n!\,n^{k-2}}{2^k}\sum_{i=0}^{n-3k}\frac{n^i}{i!}\sum_{(\lambda_3,\cdots,\lambda_p)}\frac{1}{\lambda_3!\cdots\lambda_p!} \quad (1)$$

where the second sum is taken over those partitions $(\lambda_3,\lambda_4,\cdots,\lambda_p)$ of k such that

$$\begin{cases}\lambda_3+\cdots+\lambda_p=k\\ 3\lambda_3+\cdots+p\lambda_p=n-i\end{cases}$$

The recent result obtained by Ling Jie [4] is as follows:

Theorem C Let $C^{(2)}(n,k)$ be the number of labeled connected graphs of order n with k cycles intersecting at exactly one point, then

$$C^{(2)}(n,k)=\frac{(n-1)!}{2^k}\sum_{i=0}^{n-(2k+1)}\frac{(n-i)\cdot n^i}{i!}\sum_{(\lambda_3,\cdots,\lambda_p)}\frac{1}{\lambda_3!\lambda_4!\cdots\lambda_p!} \quad (2)$$

where the second sum is taken over those partitions $(\lambda_3,\lambda_4,\cdots,\lambda_p)$ of k such that

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$$\begin{cases} \lambda_3+\lambda_4+\cdots+\lambda_p=k \\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1 \end{cases}$$

Theorem D.

$$C(n,2)=\frac{(n-1)!}{4}\sum_{i=0}^{n-5}\frac{n^i}{i!}\Big(\sum_{(\lambda)}\frac{n-i}{\lambda_3!\cdots\lambda_p!}+\sum_{(\lambda')}\frac{n}{\lambda_3!\cdots\lambda_p!}\Big) \tag{3}$$

where the second sum is taken over those partitions $(\lambda)=(\lambda_3,\lambda_4,\cdots,\lambda_p)$ of 2 such that

$$\begin{cases} \lambda_3+\lambda_4+\cdots\cdots+\lambda_p=2 \\ 2\lambda_3+3\lambda_4+\cdots+(p-1)\lambda_p=n-i-1 \end{cases}$$

the third sum is taken over those partitions $(\lambda')=(\lambda_3,\lambda_4,\cdots,\lambda_p)$ of 2 such that

$$\begin{cases} \lambda_3+\lambda_4+\cdots+\lambda_p=2 \\ 3\lambda_3+4\lambda_4+\cdots+p\lambda_p=n-i \end{cases}$$

The proceeding formulas are so complicated that application of them is very difficult. In this paper, we first obtain some simple new counting formulas which are different from formulas (1), (2), (3) and then investigate $C^{(3)}(n,k)$ for another special case. Finaly, by applying the proceeding results we shall give a counting formula for $C(n,3)$, and therefore, solve the enumeration problem of labeled connected graphs with three cycles.

Before starting our main result, the following elementary formulas are readily established.

Lemma 1[5] The number of labeled trees with n vertices $x_1,x_2,\cdots,x_n$ and and degree $d(x_i)=d_i, i=1,2,\cdots,n$ is

$$\binom{n-2}{d_1-1,\ d_2-1,\ \cdots,\ d_n-1}$$

Lemma 2[5] The number of labeled trees with n vertices $x_1,x_2,\cdots,x_n$ and degree $d(x_1)=r$ is $\binom{n-2}{r-1}(n-1)^{n-1-r}$.

Lemma 3[5] The number of Hamiltonian cycles in a complete graph K_n of order n is $(n-1)!/2$.

2. About Connected Graph With k Cycles

Theorem 1

$$C^{(1)}(n,k)=\frac{n!}{2^k k!}\sum_{p=k}^{n-2k}\frac{n^{p-2}}{(p-k)!}\binom{n-p-k-1}{k-1} \tag{4}$$

Proof Let lengths of k cycles without vertex in common each other be r_1 $r_2,\cdots,r_k$ ($r_1\geq 3$, $r_2\geq 3$, $\cdots$, $r_k\geq 3$) respectively. After each of k cycles is contracted to a vertex, k cycles become k points in a tree, $y_1,y_2,\cdots,y_k$. Let the rest of n vertices be y_{k+1}, $y_{k+2},\cdots,y_p$, then

$$p=n-r_1-r_2-\cdots-r_k+k,$$

i.e., $$r_1+r_2+\cdots+r_k=n-p+k$$
where $$k\le p\le n-2k.$$

Let degree of y_i be $d(y_i)=d_i$, $i=1,2,\cdots,p$. By Lemma 1, the number of trees with $d(y_i)=d_i$ is

$$\binom{p-2}{d_1-1,\cdots,d_p-1}$$

We consider $d(y_i)=d_i$, $i=1,2,\cdots,k$, Since y_i indicates a cycle with length r_i, there are $r_i^{d_i}$ different incidence ways of d_i edges and r_i vertices in the cycle. Thus for some groups of $d_1,d_2,\cdots,d_p\ge 1$, where $d_1+d_2+\cdots+d_p=2(p-1)$, the number of labeled graphs with n vertices and k cycles without vertex in common is

$$\binom{p-2}{d_1-1,\cdots,d_p-1}r_1^{d_1}r_2^{d_2}\cdots r_k^{d_k}.$$

Summing over all $d_1,d_2,\cdots,d_p$ we obtain

$$\sum_{\substack{d_1+d_2+\cdots+d_p=2(p-1)\\ d_1,d_2,\cdots,d_p\ge 1}}\binom{p-2}{d_1-1,\cdots,d_p-1}r_1^{d_1}\cdots r_k^{d_k}=r_1r_2\cdots r_kn^{p-2}.$$

Note that the number of all possibls k cycles with $r_1+r_2+\cdots+r_k$ vertices is

$$\frac{1}{k!}\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k}\frac{(r_1-1)!}{2}\frac{(r_2-1)!}{2}\cdots\frac{(r_k-1)!}{2}.$$

By considering different partitions of $n-p+k$, $(r_1,r_2,\cdots,r_k)$, we obtain

$$C^{(1)}(n,k)=\frac{1}{k!}\sum_{p=k}^{n-2k}\sum_{\substack{r_1+r_2+\cdots+r_k=n-p+k\\ r_i\ge 3,\ i=1,2,\cdots,k}}r_1r_2\cdots r_kn^{p-2}.$$
$$\cdot\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k}\frac{(r_1-1)!}{2}\frac{(r_2-1)!}{2}\cdots\frac{(r_k-1)!}{2}$$
$$=\frac{1}{k!}\sum_{p=k}^{n-2k}\sum_{\substack{r_1+r_2+\cdots+r_k=n-p+k\\ (r_1,r_2,\cdots,r_k\ge 3)}}\frac{n!\,n^{p-2}}{2^k(p-k)!}$$
$$=\frac{n!}{2^k\cdot k!}\sum_{p=k}^{n-2k}\frac{n^{p-2}}{(p-k)!}\sum_{\substack{x_1+x_2+\cdots+x_k=n-p-2k\\ (x_1,x_2,\cdots,x_k \text{ are nonnegative integers})}}1$$
$$=\frac{n!}{2^k\cdot k!}\sum_{p=k}^{n-2k}\frac{n^{p-2}}{(p-k)!}\binom{n-p-2k+k-1}{n-p-2k}$$
$$=\frac{n!}{2^k\cdot k!}\sum_{p=k}^{n-2k}\frac{n^{p-2}}{(p-k)!}\binom{n-p-k-1}{k-1}.$$

As a verification, applying formula (4) with $k=1$, we obtain Renyi's formula

$$C(n,1)=\frac{1}{2}n!\sum_{p=1}^{n-2}\frac{n^{p-2}}{(p-1)!}\binom{n-p-l-1}{0}=\frac{1}{2}\sum_{p=3}^{n}[n-1]_{p-1}n^{n-p}$$

Clearly, our result is better than Tomescu's formula (1).

Theorem 2

$$C^{(2)}(n,k)=\frac{n!}{2^kk!}\sum_{p=1}^{n-2k}\frac{(n+1-p)n^{p-2}}{(p-1)!}\binom{n-p-k-1}{k-1}\quad(k\ge 2)\qquad(5)$$

Proof Let lengths of k cycles intersecting at a common point be $r_1, r_2, \cdots, r_k$ ($r_1, r_2, \cdots, r_k \geq 3$) respectively. And let the total number of vertices of k cycles be p. Then

$$r_1 + r_2 + \cdots + r_k - k + 1 = p$$

and
$$2k+1 \leq p \leq n.$$

By Clarke's formula (see Lemma 2), the number of trees with vertices $x_0, x_{p+1}, x_{p+2}, \cdots, x_n$ and $d(x_0) = r$ is

$$\binom{n-p-1}{r-1}(n-p)^{n-p-r}, \quad r = 1, 2, \cdots, n-p.$$

If vertex x_0 is replaced by k cycles intersecting at a common point, then there are p^r different incidence ways of r edges and k cycles. Thus for a group of $(r_1, r_2, \cdots, r_k)$, $r_1 + r_2 + \cdots + r_k - k + 1 = p$, the number of labeled connected graphs of order n with k cycles intersecting at a common point is

$$\sum_{r=1}^{n-p} p^r \binom{n-p-1}{r-1}(n-p)^{n-p-r} = pn^{n-p-1}$$

Note that all possible cases of k cycles ($k \geq 2$) correspond to the partition $r_1 + r_2 + \cdots + r_k = p + k - 1$, $r_1, r_2, \cdots, r_k \geq 3$. Summing from $p = 2k+1$ to n, we obtain

$$C^{(2)}(n, k) = \sum_{p=2k+1}^{n} \frac{1}{k!} \sum_{\substack{r_1+r_2+\cdots+r_k = p+k-1 \\ r_i \geq 3,\ i=1,2,\cdots,k}} pn^{n-p-1}\binom{n}{p}\binom{p}{1}\binom{p-1}{r_1-1}$$

$$\cdot \binom{p-r_1}{r_2-1} \cdots \binom{r_k-1}{r_k-1} \cdot \frac{(r_1-1)!}{2} \cdot \frac{(r_2-1)!}{2} \cdots \frac{(r_k-1)!}{2}$$

$$= \frac{n!}{2^k k!} \sum_{p=2k+1}^{n} \frac{pn^{n-p-1}}{(n-p)!} \sum_{\substack{x_1+x_2+\cdots+x_k = p-2k-1 \\ (x_1, x_2, \cdots, x_k \text{ are nonnegative integers})}} 1$$

$$= \frac{n!}{2^k k!} \sum_{p=2k+1}^{n} \frac{pn^{n-p-1}}{(n-p)!}\binom{p-k-2}{k-1}$$

$$= \frac{n!}{2^k k!} \sum_{p=1}^{n-2k} \frac{(n+1-p)n^{p-2}}{(p-1)!}\binom{n-p-k-1}{k-1}$$

Obviously, formula (5) is better than formula (3) because our result avoids the enumeration of partitions.

Now we consider another spacial case about connected graphs with k cycles. Let $C^{(3)}(n, k)$ be the number of labeled connected graphs of order n with k cycles which has a vertex in common one by one as a chain (whose block-cutpoint-graph is a path). We have

Theorem 3

$$C^{(3)}(n, k) = \frac{n!}{2^{k+1}} \sum_{p=2k+1}^{n} \frac{pn^{n-p-1}}{(n-p)!} \sum_{\substack{r_1+r_2+\cdots+r_k = p+k-1 \\ r_1, r_2, \cdots, r_k \geq 3}} (r_2-1)(r_3-1)\cdots(r_{k-1}-1) \quad (6)$$

Proof Let the length of k cycles be $r_1, r_2, \cdots, r_k$ respectively and the total number of vertices of k cycles be p. Then

$$r_1+r_2+\cdots+r_k=p+k-1$$
$$2k+1\leq n. \qquad p\leq n.$$

By Clarke's formula (see Lemma 2), the number of trees with vertices $x_0, x_{p+1}, x_{p+2}, \cdots, x_n$ and $d(x_0)=r$ is $\binom{n-p-1}{r-1}(n-p)^{n-p-r}$, $r=1,2,\cdots,n-p$. After vertex x_0 is replaced by a cycle of length r, there are p^r different incidence ways of r edges and r-cycle. Thus for a group of $(r_1, r_2, \cdots, r_k)$, $r_1+\cdots+r_k=p+k-1$, the number of graphs is

$$\sum_{r=1}^{n-p} p^r\binom{n-p-1}{r-1}(n-p)^{n-p-r}=pn^{n-p-1}.$$

Note that the number of all possible k cycles with p vertices is

$$\binom{n}{p}\binom{p}{k-1}(k-1)!\binom{p-k+1}{r_1-1}\binom{p-k-r_1+2}{r_2-2}\cdots\binom{r_k-1}{r_k-1}$$
$$\cdot\frac{(r_1-1)!}{2}\cdot\frac{(r_3-1)!}{2}\cdots\frac{(r_k-1)!}{2}.$$
$$=\frac{n!}{2^k}\cdot\frac{(r_2-1)(r_3-1)\cdots(r_{k-1}-1)}{(n-p)!}$$

Summing from $p=2k+1$ to n, we obtain

$$C^{(3)}(n,k)=\sum_{p=2k+1}^{n} pn^{n-p-1}\sum_{\substack{r_1+r_2+\cdots+r_k=p+k-1\\(r_i\geq 3,\ i=1,2,\cdots,k)}}\frac{n!(r_2-1)(r_3-1)\cdots(r_{k-1}-1)}{2^k(n-p)!}$$
$$=\frac{n!}{2^{k+1}}\sum_{p=2k+1}^{n}\frac{pn^{n-p-1}}{(n-p)!}\sum_{\substack{r_1+r_2+\cdots+r_k=p+k-1\\(r_i\geq 3,\ i=1,2,\cdots,k)}}(r_2-1)(r_3-1)\cdots(r_{k-1}-1)$$

The following result will be useful later.

Corollary 3.1

$$C^{(3)}(n,3)=\frac{n!}{96}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p^3-12p^2+41p-30) \qquad (7)$$

Proof By formula (6) $k=3$,

$$C^{(3)}(n,3)=\frac{n!}{2^4}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}\sum_{\substack{k_1+k_2+k_3=p+2\\(k_1,k_2,k_3\geq 3)}}(k_2-1)$$
$$=\frac{n!}{2^4}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}\sum_{k_2=3}^{p-4}(k_2-1)\sum_{\substack{k_1+k_3=p+2-k_2\\(k_1,k_3\geq 3)}}$$
$$=\frac{n!}{2^4}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}\sum_{k_2=3}^{p-4}(k_2-1)(p-k_2-3)$$
$$=\frac{n!}{2^4}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}\cdot\frac{1}{6}(p^3-12p^2+41p-30)$$
$$=\frac{n!}{96}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p^3-12p^2+41p-30)$$

3. New Counting Formula for $C(n,2)$

Applying the proceeding results we obtain a new counting formula for $C(n,$

2) which is better than formula (3).

Theorem 4

$$C(n,2)=\frac{n!(n-4)}{8}+\frac{n!}{8}\sum_{p=2}^{n-4}\frac{n^{p-1}}{(p-1)!}(n-p-3) \qquad (8)$$

Proof It is not difficult to see that

$$C(n,2)=C^{(1)}(n,2)+C^{(2)}(n,2)$$

By theorems 1 and 2,

$$C(n,2)=\frac{n!}{8}\sum_{p=2}^{n-4}\frac{n^{p-2}}{(p-2)!}(n-p-3)+\frac{n!}{8}\sum_{p=5}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p-4)$$

$$=\frac{n!}{8}\left[\sum_{p=2}^{n-4}\frac{n^{p-2}}{(p-2)!}(n-p-3)+\sum_{p=1}^{n-4}\frac{(n+1-p)n^{p-2}}{(p-1)!}(n-p-3)\right]$$

$$=\frac{n!}{8}\left[n-4+\sum_{p=2}^{n-4}\frac{n^{p-2}}{(p-2)!}(n-p+3)\left(1+\frac{n+1-p}{p-1}\right)\right]$$

$$=\frac{n!}{8}(n-4)+\frac{n!}{8}\sum_{p=2}^{n-4}\frac{n^{p-2}}{(p-1)!}n(n-p-3)$$

$$=\frac{n!(n-4)}{8}+\frac{n!}{8}\sum_{p=2}^{n-4}\frac{n^{p-1}}{(p-1)!}(n-p-3)$$

Formula (8) avoids the enumeration of partitions.

4 . The Counting Formula for $C(n,3)$

For all labeled connected grabhs of order n with exactly three cycles, we consider the following three cases.

Case 1 Three cycles have no vertex in common. Let the number of corresponding graphs be $C_1(n,3)$.

According to Theorem 1.

$$C_1(n,3)=C^{(1)}(n,3)=\frac{n!}{2^3\cdot 3!}\sum_{p=3}^{n-6}\frac{n^{p-2}}{(p-3)!}\binom{n-p-4}{2}$$

$$=\frac{n!}{96}\sum_{p=3}^{n-6}\frac{n^{p-2}}{(p-3)!}(n-p-4)(n-p-5).$$

Case 2 There are exactly two cycles with a vertex in common. Let the number of corresponding graphs be $C_2(n,3)$.

Suppose that the number of vertices of two cycles intersecting at a point is $k_1(k_1\geq 5)$, the number of the remaining cycle is $k_3(k_2\geq 3)$.

Let the preceeding k_1 vertices (k_2 vertices) be contracted to a point denoted by $x_1(x_2)$. Other vertices are denoted by x_i, $3\leq i\leq n-k_1-k_2+2$. By Lemma 1, the number of trees of order $p=n-k_1-k_2+2$ $(2\leq p\leq n-6)$ with $d(x_i)=d_i$, $i=1,2,\cdots,p$, is

$$\binom{p-2}{d_1-1,\cdots,d_p-1}$$

Using similar argument to theorem 1 for all $(d_1,d_2,\cdots,d_p)$, $d_i\geq 1$, where

$d_1+d_2+\cdots+d_p=2(p-1)$, the number of graphs is

$$\sum_{\substack{d_1+d_2+\cdots+d_p=2(p-1)\\ d_1,d_2,\cdots,d_p\geq 1}} \binom{p-2}{d_1-1,\cdots,d_p-1} k_1^{d_1}k_2^{d_2}=k_1k_2n^{p-2}.$$

It is not difficult to get that the number of two cycles with k_1 vertices, intersecting at a common point, is

$$\frac{1}{2}\sum_{i=2}^{k_1-3} k_1\binom{k_1-1}{i}\frac{i!}{2}\cdot\frac{(k_1-i-1)!}{2}$$
$$=\frac{1}{8}\sum_{i=2}^{k_1-3}k_1!=\frac{1}{8}k_1!(k_1-4),\qquad (k_1\geq 5).$$

Thus

$$C_2(n,2)=\sum_{p=2}^{n-6}\sum_{\substack{k_1+k_2=n-p+2\\ k_1\geq 5,\ k_2\geq 3}} k_1k_2n^{p-2}\binom{n}{k_1}\binom{n-k_1}{k_2}\cdot\frac{1}{8}k_1!(k_1-4)\cdot\frac{(k_2-1)!}{2}$$
$$=\frac{n!}{16}\sum_{p=2}^{n-6}\frac{n^{p-2}}{(p-2)!}\sum_{k_1=5}^{n-p-1}(k_1^2-4k_1)$$
$$=\frac{n!}{96}\sum_{p=2}^{n-6}\frac{n^{p-2}}{(p-2)!}[(n-p-1)(n-p)(2n-2p-13)+60].$$

Case 3 Three cycles intersect as a connected component. Let the number of corresponding graphs be $C_3(n,3)$.

Consider the following three subcases.

Subcase 3.1 three cycles are a chain.

By Corollary 3.1, the required number is

$$C^{(3)}(n,3)=\frac{n!}{96}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p^3-12p^2+41p-30) \tag{9}$$

Subcase 3.2 Three cycles intersect at exactly one point.

By Theorem 2, the required number is

$$C^{(2)}(n,3)=\frac{n!}{96}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}\binom{p-5}{2} \tag{10}$$
$$=\frac{n!}{96}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p^2-11p+30). \tag{10}$$

Subcase 3 two cycles have a path of length $l\geq 1$ in common to produce three cycles.

Actually, these three cycles consist of three paths with two endpoints in common. Let the lengths of the preceeding three paths be k_1, k_2, k_3 respectively, and the total number of vertices of three paths be p. Then

$$k_1+k_2+k_3=p+1,\ 4\leq p\leq n,$$

$k_1\geq 1$, $k_2\geq 1$, $k_3\geq 1$ and only one of k_1, k_2, k_3 is allowed to be 1.

The number of such three cycles with p vertices is

$$\frac{1}{6}\sum_{\substack{k_1+k_2+k_3=p+1\\ (k_1,k_2,k_3\geq 1)}}\binom{p}{2}\binom{p-2}{k_1-1}\binom{p-k_1-1}{k_2-1}(k_1-1)!(k_2-1)!(k_3-1)!-\frac{1}{2}\cdot\frac{p(p-1)!}{2}$$

$$=\frac{1}{6}\sum_{\substack{x_1+x_2+x_3=p-2\\(x_1,\,x_2,\,x_3\text{ are nonnegative integers})}}\cdot\frac{p!}{2}-\frac{p(p-1)!}{4}$$

$$=\frac{p!}{12}\binom{p}{2}\frac{p(p-1)!}{4}=\frac{p!}{24}p(p-1)-\frac{p(p-1)!}{4}$$

$$=\frac{p!}{24}(p-3)(p+2).$$

Hence, using similar argument to theorem 1, the number of the required graphs is

$$\sum_{p=4}^{n}pn^{n-p-1}\binom{n}{p}\frac{p!}{24}(p-3)(p+2)$$

$$=\frac{n!}{24}\sum_{p=4}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p-3)(p+2)$$

Summing results (9), (10), (11), we have

$$C_3(n,3)=\frac{n!}{96}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p^3-12p^2+41p-30)$$

$$+\frac{n!}{96}\sum_{p=7}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p^2-11p+30)$$

$$+\frac{n!}{24}\sum_{p=4}^{n}\frac{pn^{n-p-1}}{(n-p)!}(p^2-p-6)$$

$$=\frac{n!}{96}\sum_{p=0}^{n-7}\frac{(n-p)^2n^{p-1}}{p!}(n-p-5)(n-p-6)$$

$$+\frac{n!}{24}\sum_{p=0}^{n-4}\frac{(n-p)n^{p-1}}{p!}(n-p-3)(n-p+2).$$

Thus we obtain

Theorem 5 The number of labelad connected graphs of order n with 3 cycles is

$$C(n,3)=\frac{n!}{96}\sum_{p=0}^{n-9}\frac{n^{p+1}}{p!}(n-p-7)(n-p-8)$$

$$+\frac{n!}{96}\sum_{p=0}^{n-8}\frac{n^p}{p!}[(n-p-3)(n-p-2)(2n-2p-17)+60]$$

$$+\frac{n!}{96}\sum_{p=0}^{n-7}\frac{(n-p)^2n^{p-1}}{p!}(n-p-5)(n-p-6)$$

$$+\frac{n!}{24}\sum_{p=0}^{n-4}\frac{(n-p)n^{p-1}}{p!}(n-p-3)(n-p+2).$$

Proof $C(n,3)=C_1(n,3)+C_2(n,3)+C_3(n,3).$

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关于含 k 圈标号图的计数问题

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含 k 个圈的标号图的计数问题是一个未解决问题．迄今仅对于 $k=1,2$ 被解决，可是，所得出的计数式均较复杂．本文改进了已得到的一系列公式，并且解决了 $k=3$ 的上述计数问题．