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The Behrens Radical of a Full Matrix Ring over an Associative Ring*

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The following problem is posed by F. A. Szasz [2 p208] "Investigate the Behrens radical of a full matrix ring."

In this note we consider this problem. We recall that the Behrens radical class is the upper radical determined by the class of all subdirectly irreducible rings which contain a non-zero idempotent element in their heart. We prove that $J_B(R_n)\subseteq(J_B(R))_n$ holds for every ring R. We then construct a simple ring R to show that in R the equality does not hold generally.

Theorem 1 *A ring R is irreducible with idempotent heart if and only if R_n is irrducible with idempotent heart.*

Proof Let R be an irreducible ring with idempotent heart H. Let Q be any non-zero ideal of R_n. and let P be the ideal of R generated by the elements of matrices in Q. Obviously, H is contained in P. Since

$$P_n^3\subseteq Q\subseteq P_n$$

we have

$$H_n^2=H_n\ ,\ H_n=H_n^3\supseteq P_n^3\subseteq Q$$

Thus H_n is the idempotent heart of R_n.

Conversely Let R_n be an irreducible ring with idempotent heart Q. Let H be the ideal of R generated by the elements of matrices in Q. Then we have,

$$H_n^3\subseteq Q\subseteq H_n$$

thus, $Q=Q^3\subseteq H_n^3\subseteq Q$. Hence, $Q=(H^3)_n$, and H^3 is a non-nilpotent ideal of R.

For any non-zero ideal I of R, we have $Q\subseteq I_n$, i.e., $(H^3)_n\subseteq I_n$. Hence H^3 is contained in I. Therefore, H^3 is the idempotent heart of R.

Theorem 2 *For every ring R, the Behrens radical of R_n is contained in the full matrix ring of $J_B(R)$, i.e., $J_B(R_n)\subseteq(J_B(R))_n$. Let R be a ring, $J_B(\overline{R_n})=(J_B(\overline{R}))_n$ holds for every homomorphic image $\overline{R}$ of R if and only if the following condition is satisfied: for any irreducible homomorphic image R_1 of R, there is no non-zero idempotent element in H_n, if there is no non-zero idempotent*

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element in H, where H is the heart of R_1.

Proof For every ring R, the Behrens radical of R is the intersection of all those ideals P_a of R, for which R/P_a is irreducible whose heart containing a non-zero idempotent element. Then from Theorem 1, we conclude that if a ring R is irreducible with heart containing a non-zero idempotent element, then R_n is irreducible with the same property. Thus we know from

$$(R/(P_a))_n = R_n/(P_a)_n$$

that the following inclusion holds for any ring R:

$$J_B(R_n) \subseteq (J_B(R))_n.$$

To prove the second part of this theorem, first we suppose that there is an irreducible homomophic image R_1 of R with hear H. Suppose there is no non-zero idempotent element in H, but there is such an element in H_n. Then we have

$$J_B(R_{1n}) = 0, \quad (J_B(R_1))_n \neq 0,$$

since H is a J_B-radical ring. Then we have $J_B(R_{1n}) \neq (J_B(R_1))_n$.

Conversely Suppose $\overline{R}$ is a homomorphic image of R. Let Q be any ideal of $\overline{R}_n$ such that $\overline{R}_n/Q$ is irreducible with heart containing a non-zero idempotent element. Let P be the ideal of $\overline{R}$ generated by the elements of matrices in Q. Then we have

$$P_n^3 \subseteq Q \subseteq P_n$$

Since $\overline{R}_n/Q$ is a semi-prime ring, we have $P_n \subseteq Q$. Thus $P_n = Q$.

From theorem 1 and

$$(\overline{R}/P)_n = \overline{R}_n/Q$$

we know that $\overline{R}/P$ is an irreducible image of R with idempotent heart H. Then H_n is the heart of $\overline{R}_n/Q$ and therefore there is a non-zero idempotet element in H_n. From the conditions given, we know that there exists a non-zero idempotent element in H. Thus we have proved the following result:

Q is an ideal of $\overline{R}_n$ such that $\overline{R}_n/Q$ irreducible with heart containing a non-zero idempotent element if and only if there exists an ideal P of $\overline{R}$ such that $Q = P_n$ and $\overline{R}/P$ irreducible with heart containing a non-zero idempotent element. Thus, for any ring homomorphic image $\overline{R}$ of R, we have

$$J_B(\overline{R}_n) = (J_B(\overline{R}))_n.$$

This completes the proof.

Now we construct a simple ring R for which $J_B(R_n) \neq (J_B(R))_n$.

Example 1 Let A be the set of all such elements:

$$\sum_{0 \leq m < \infty} x^m a_m$$

where x is an indeterminate and a_m is a rational function of y with real coefficients. We define

$\sum x^m a_m = \sum x^n b_n$, if $a_k = b_k$ for all integers k

$$\sum_{0\le m<\infty} x^m a_m + \sum_{0\le n<\infty} x^n b_n = \sum_{0\le m<\infty} x^m (a_m + b_m) \quad ,$$

we define multiplication in A by distributive law and the following:

$$a(y)x^n = x^n a(y) + x^{n-1} c_n^1 a^{(1)}(y) + \cdots + c_n^n a^{(n)}(y),$$

where $a^{(k)}(y)$ denotes the k'th derivation of $a(y)$ for y. Then we can verilfy that A is an associative simple ring with unique non-zero idempotent element. Also, there is no divisor of zero in A.

Now let $H = Ay$. Since

$$H^2 = (Ay)(Ay) = (AyA)y = Ay = H,$$

H is idempotent. To prove that H is a simple ring, let I be any non-zero ideal of H. For any non-zero element h of I, since $yh \neq 0$, we have

$$H = Ay = A(yh)Ay = (Ay)h(Ay) = HhH \subseteq I$$

Thus $H = I$. Hence H is simple without non-zero idempotent element.

For any natural number $n \geq 2$, there exists a non-zero idempotent in H_n. A direct calculation shows that the following element is idempotent:

$$\begin{bmatrix} xy & x^2y & 0 & \cdots & 0 \\ -y & -xy & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Thus we have the following formula:

$$J_B(H_n) = 0, \quad (J_B(H))_n = H_n.$$

Hence we obtain a non-trivial inclusion $J_B(H_n) \subset (J_B(H))_n$.

References

[1] N. J. Divinsky, Rings and Redicals, Univ. of Toronto Press, 1990.

[2] F. A. Szasz, Radicals of Rings, A Wiley-Interscience Publication, 1981.

结合环上全矩阵环的Behrens根

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Szasz在［2］中建议考察结合环上全矩阵环的Behrens根，本文完全解决了这一公开问题.

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