Journal of Mathematical Research and Exposition
Vol. 11 No.4, Nov., 1991

Derivations in A Prime Γ-Ring*

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Abstract

In this paper, we extend Posner's theorem and the nilpotency of a derivation to a prime Γ-ring. Our main result is the following theorem: Let R be a prime Γ-ring with two derivations d, f, char $R\neq 2$. If df is a derivation too, then either $d=0$ or $f=0$.

The notion of a prime Γ-ring was introduced by Luh [1]. In this paper, we define a derivation in a prime Γ-ring and give a generalization of Posner's theorem and some nilpotent properties for a prime Γ-ring with a derivation.

Let R be a prime Γ-ring. A derivation d on R is a mapping $d: R\to R$ satisfying the condition that:

$$\mathrm{d}(x+y)=\mathrm{d}(x)+\mathrm{d}(y)$$
$$\mathrm{d}(x\alpha y)=\mathrm{d}(x)\alpha y+x\alpha\mathrm{d}(y),\quad \text{for any } x,y\in R,\ \alpha\in\Gamma.$$

Lemma 1 Let R be a prime Γ-ring with a derivation d. $0\neq a\in R$. If for any $x\in R$, $\alpha\in\Gamma$, $a\alpha\mathrm{d}(x)=0$, then $\mathrm{d}(R)=0$.

Proof For any $x,y\in R$, we have

$$\begin{aligned} a\alpha\mathrm{d}(x\beta y)&=a\alpha(\mathrm{d}(x)\beta y+x\beta\mathrm{d}(y))\\ &=a\alpha\mathrm{d}(x)\beta y+a\alpha x\beta\mathrm{d}(y)\\ &=a\alpha x\beta\mathrm{d}(y)=0\end{aligned}$$

By the primeness of R and $a\neq 0$ we obtain that $\mathrm{d}(y)=0$, for all $y\in R$.

Posner[2] proved the following theorem: Let R be a prime ring with two derivations d, f, char $R\neq 2$. If df is also a derivation then either $d=0$, or $f=0$. In following theorem Posner's theorem has been proved in a prime Γ-ring.

Theorem 2 Let R be a prime Γ-ring with two derivations d, f, char $R\neq 2$. If df is a derivation too, then either $d=0$ or $f=0$.

Proof For any $x,y\in R$, $\alpha\in\Gamma$, we have

$$\begin{aligned}\mathrm{d}f(x\alpha y)&=\mathrm{d}(f(x)\alpha y+x\alpha f(y))\\ &=\mathrm{d}f(x)\alpha y+f(x)\alpha\mathrm{d}(y)+\mathrm{d}(x)\alpha f(y)+x\alpha\mathrm{d}f(x)\end{aligned}$$
$$\mathrm{d}f(x\alpha y)=\mathrm{d}f(x)\alpha y+x\alpha\mathrm{d}f(y).$$

* Received Jan. 3, 1990.

Then $$f(x)\alpha d(y)+d(x)\alpha f(y)=0 \quad (1)$$

Let $\beta\in\Gamma$ and replace x by $y\beta d(x)$ in (1), we get

$$f(y\beta d(x))\alpha d(y)+d(y\beta d(x))\alpha f(y)=0$$

Thas is:

$$f(y)\beta d(x)\alpha d(y)+d(y)\beta d(x)\alpha f(y)+y\beta(fd(x)\alpha d(y)+d^2(x)\alpha f(y))=0$$

Since $fd(x)\alpha d(y)+d^2(x)\alpha f(y)=0$, so

$$f(y)\beta d(x)\alpha d(y)+d(y)\beta d(x)\alpha f(y)=0 \quad (2)$$

From (1), we have $f(x)\alpha d(y)=-d(x)\alpha f(y)$. Then

$$(f(y)\beta d(x)-d(y)\beta f(x))\alpha d(y)=0$$

By Lemma 1, either $f(y)\beta d(x)-d(y)\beta f(x)=9$ or $d(y)=0$. If $f(y)\beta d(x)-d(y)\beta f(x)=0$ from (1) it follows that $2f(y)\alpha d(x)=0$. Because char $R\neq 2$ and R is prime,

$$f(y)\alpha d(x)=0 \quad (3)$$

Let $z\in R$, $\beta\in\Gamma$ and replace y by $y\beta z$ in (3), we have $f(y\beta z)\alpha d(x)=0$, i.e.,

$$f(y)\beta z\alpha d(x)+y\beta f(z)\alpha d(x)=0$$

Hence $f(y)\beta z\alpha d(x)=0$. By the primeness of R we obtain that either $f(y)=0$ or $d(x)=0$, for any x, $y\in R$. This completes the proof.

Next we shall consider some nilpotent properties of derivation in a prime Γ-ring. At frist we can obtain the Leibniz' rule of a prime Γ-ring with a derivation by induction:

$$d^n(x\alpha y)=\sum_{i=0}^{n}\binom{n}{i}d^{n-i}(x)\alpha d^i(y)$$

Lemma 3 A Γ-ring R is a prime Γ-ring if and only if if $I_1\Gamma I_2=\{0\}$ then $I_1=\{0\}$ or $I_2=\{0\}$, where I_1, I_2 are two left (right) ideals of R.

Proof The sufficiency follows from the definition of a prime Γ-ring.

Conversely, let R be a prime Γ-ring with two left ideals I_1, I_2 such that $I_1\Gamma I_2=\{0\}$. Then

$$(I_1\Gamma R)\Gamma(I_2\Gamma R)=(I_1\Gamma)(R\Gamma I_2)(\Gamma R)\subseteq(I_1\Gamma)I_2(\Gamma R)=\{0\}\Gamma R=\{0\}.$$

Since $I_1\Gamma R$, $I_2\Gamma R$ are two-sided ideals of R, it follows that either $I_1\Gamma R=\{0\}$ or $I_2\Gamma R=\{0\}$. If $I_1\Gamma R=\{0\}$ then $I_1\Gamma R\subseteq I_1$ i.e., I_1 is a two-sided ideal of R. By the primeness of R we obtain that $I_1=\{0\}$. Similarly if $I_2\Gamma R=\{0\}$ then $I_2=\{0\}$.

Lemma 4 Let R be a prime Γ-ring. If I is a two-sided ideal of R then I is a prime Γ-ring.

Proof Let $a\Gamma I\Gamma b=\{0\}$, $a, b\in I$. Then

$$(a\Gamma R)\Gamma(b\Gamma R)\Gamma(b\Gamma R)=a\Gamma(R\Gamma b\Gamma R)\Gamma(b\Gamma R)$$
$$\subseteq a\Gamma I\Gamma(b\Gamma R)=(a\Gamma I\Gamma b)\Gamma R=\{0\}\Gamma R=\{0\}.$$

From Lemma 3, we obtain that either $a\Gamma R=\{0\}$ or $b\Gamma R=\{0\}$. Therefore $a\Gamma R\Gamma a=\{0\}$ or $b\Gamma R\Gamma b=\{0\}$. However R is a prime Γ-ring, then $a=0$ or $b=0$.

Theorem 5 Let d be a derivation of a prime Γ-ring R. I be a nonzero ideai of R, n is a positive integer. If $d^n(I)=0$ then $d^{2n-1}(R)=0$.

Proof Let $J=I+d(I)+\cdots+d^{n-1}(I)$. Clearly $d(J)\subseteq J$. We can prove that J is a nonzero ideal of R. For any $x\in R$, $a\in\Gamma$, $y=y_0+d(y_1)+\cdots+d^{n-1}(y_{n-1})\in J$, $y_i\in I$, $i=0,1,\cdots,n-1$. We have

$$y_i a x\in I\subseteq J,\quad i=0,1,\cdots,n-1.$$

$$d(y_1)ax=d(y_1ax)-y_1ad(x)\in J,$$

$$d^2(y_2)ax=d^2(y_2ax)-d(y_2)ad(x)-d(y_2ad(x))$$

$$=d^2(y_2ax)-(d(y_2ad(x))-y_2ad^2(x))-d(y_2ad(x))\in J,$$

By induction we can obtain that $d^{n-1}(y_{n-1})ax\in J$. So

$$yax=(y_0+d(y_1)+\cdots+d^{n-1}(y_{n-1}))ax$$

$$=y_0ax+d(y_1)ax+\cdots+d^{n-1}(y_{n-1})ax\in J.$$

Similarly $xay\in J$. Hence J is a nonzero ideal of R, and $d^n(J)=0$. Let $y\in J, a\in\Gamma$, $r\in R$. By Leibniz' rule

$$0=d^n(d^{n-1}(y)ar)=\sum_{i=0}^{n}\binom{n}{i}d^{n-i}(d^{n-1}(y))ad^i(r)=d^{n-1}(y)ad^n(r)$$

Similarly, we have

$$0=d^n(d^{n-2}(y)ad(r))=\sum_{i=0}^{n}\binom{n}{i}d^{n-i}(d^{n-2}(y))ad^i(d(r))=d^{n-2}(y)ad^{n+1}(r)$$

Continuing this argument we finally reach the identity $0=yad^{2n-1}(r)$ for all $y\in J$ and $a\in\Gamma$. Set $y=x\beta y$, $x\in R$ $\beta\in\Gamma$, then $0=x\beta yad^{2n-1}(r)$ for all $r\in R$. By the primeness of J we obtain that $d^{2n-1}(r)=0$ for all $r\in R$. This completes the proof.

Lemma 6 Let R be a prime Γ-ring with a derivation d, I be a nonzero ideal of R. If there exists a positive integer n, and $a\in R$ such that for any $x\in I$, $\alpha\in\Gamma$, $a\alpha d^n(x)=0$. Then $a=0$ or $d^{3n-1}(R)=0$.

Proof Let the set $J=I+d(I)+\cdots+d^n(I)+\cdots$ consist of all elements of R that can be written as a finite sum $y=y_0+d^{i_1}(y_{i_1})+\cdots+d^{i_n}(y_{i_n})$, where n depends on y, $i_j\in N, y_{i_j}\in I$, $j=1,2,\cdots,n$. We can prove that J is a nonzero ideal of R, and $d(J)\subseteq J$, $a\alpha d^n(J)=0$. For any $x\in J$, $y\in R$, $\alpha,\beta\in\Gamma$, we have

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Continuing this argument we finally reach the identity

$$\alpha\beta x a d^{3n-1}(y)=0, \text{ for any } x\in J,\ y\in R$$

By the primeness of J, either $a=0$ or $d^{3n-1}(R)=0$.

Theorem 7 Let R be a prime Γ-ring with two derivations d, f, I be a nonzero ideal of R. For a positive integer $n>1$, if for any $x\in I$, $d_1d_2^n(x)=0$ then $d_1(R)=0$ or $d_2^{3n-1}(R)=0$.

Proof Similar to Lemma 6. Let $J=I+d(I)+d^2(I)+\cdots$ then J is a nonzero ideal of R, and $d_1d_2^n(J)=0$. Let $x\in J$, $y\in R$, $a\in\Gamma$, then

$$d_1d_2^n(xay)=d_1\Big(\sum_{i=0}^{n}\binom{n}{i}d_2^i(x)ad_2^{n-i}(y)\Big)=0 \tag{1}$$

Replacing x, y by $d_2^{n-1}(x)$, $d_2^n(y)$ in (1) we have

$$d_1\Big(\sum_{i=0}^{n}\binom{n}{i}d_2^{n+i-1}(x)ad_2^{2n-i}(y)\Big)=d_1(d_2^{n-1}(x)ad_2^{2n}(y))=0$$

Replacing x, y by $d_2^{n-2}(x)$, $d_2^{n+1}(y)$ in (1) and using above result we get $d_1(d_2^{n-2}(x)ad_2^{2n+1}(y))=0$. Continuing this argument we finally reach the identity

$$d_1(x)ad_2^{3n-1}(y)=0 \text{ for any } x\in J,\ y\in R,\ a\in\Gamma$$

By Lemma 6, either $d_1(J)=0$ or $d_2^{3n-1}(R)=0$. From $d_1(J)=0$ it follows that for any $x\in J$, $y\in R$, $a\in\Gamma$ $d_1(xay)=0$ i.e. $d_1(xay)=d_1(x)ay+xad_1(y)=xad_1(y)=0$. Since R is a prime Γ-ring, hence $d_1(y)=0$, for all $y\in R$. This completes the proof.

This theorem give a generalization of Posner's theorem.

Acknowledgements: The author wishes to thank Prof. Xie Bangjie and Prof. Niu Fengwen for their advice and encouragement.

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